Stability of the 2-D Givone-Roesser Model with Periodic Coefficients

T. Bose*, M.-Q. Chen**, and R. Thamvichai

Abstract—The stability of two-dimensional (2-D) periodically shift varying (PSV) filters formulated as the Givone-Roesser (GR) model is studied. The applications of these filters include processing video signals with cyclostationary noise, image and video scrambling, and design of multiplierless filters. The GR model is embedded into the second model of Fornasini-Marchesini (FM) and the stability of this embedded model is then studied. Several sufficient conditions and one necessary condition are derived for stability. These conditions are compared for their relative computational complexities and their restrictions. Based on the computational complexities of implementing these conditions, an algorithm is proposed to determine the stability of a given 2-D PSV system. Several examples are given to illustrate the results.

I. INTRODUCTION

Two dimensional (2-D) periodically shift varying (PSV) filters have received widespread attention for the past few decades because of their applications in various fields. Image scrambling, processing images with cyclostationary noise and the design, analysis and implementation of 2-D multirate filter banks are applications of 2-D PSV filters. However, little attention has been given to the analysis of 2-D PSV filters.

In [1], 2-D PSV filters have been analyzed in direct form. Equivalent shift-invariant block structures were derived for 2-D state-space PSV filters in [2]. These structures are useful for analysis but are computationally inefficient for implementation. Other shift-invariant structures for 2-D PSV systems were proposed in [3],[4] and analyzed for stability. The stability of 2-D PSV filters have been studied to some extent. In [5], 2-D PSV filters formulated as the second model of Fornasini-Marchesini were considered for stability and some conditions and properties were established. Some other references on linear multidimensional systems include [6]-[9].

In this paper, 2-D PSV filters are formulated as a Givone-Roesser model with periodic coefficients, and studied for stability. Several sufficient conditions and one necessary conditions are established. Based on the computational complexities of implementing these conditions, an algorithm is proposed to determine the stability of a given 2-D PSV filter. Several examples are provided to illustrate the different cases that can arise in the algorithm.

II. SYSTEM DESCRIPTION

The difference equation for a linear 2-D PSV system can be written as

\[
y(i, j) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} a_{mn}(i, j) y(i - m, j - n) - \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} b_{mn}(i, j) u(i - m, j - n)
\]  

where \((m, n) \neq (0, 0)\) for \(a_{mn}\). The coefficients are periodically shift variant with period \((P, Q)\) where \(P, Q\) are positive integers, not both zero, i.e. \(a_{mn}(i, j) = a_{mn}(i + P, j) = a_{mn}(i, j + Q)\) and \(b_{mn}(i, j) = b_{mn}(i + P, j) = b_{mn}(i, j + Q)\).

As with a 2-D LSIV system, several different state-space forms with periodic coefficient matrices can be used to represent the above 2-D PSV system. The Givone-Roesser (GR) model [11] is given by:

\[
\begin{bmatrix}
  x^h(i+1, j) \\
  x^v(i, j+1)
\end{bmatrix} =
\begin{bmatrix}
  A_1(i, j) & A_2(i, j) \\
  A_3(i, j) & A_4(i, j)
\end{bmatrix}
\begin{bmatrix}
  x^h(i, j) \\
  x^v(i, j)
\end{bmatrix}
+ \begin{bmatrix}
  B_1(i, j) \\
  B_2(i, j)
\end{bmatrix} u(i, j)
\]

\[
y(i, j) = \begin{bmatrix}
  C_1(i, j) \\
  C_2(i, j)
\end{bmatrix}
\begin{bmatrix}
  x^h(i, j) \\
  x^v(i, j)
\end{bmatrix}
+ D(i, j) u(i, j)
\]  

where \(x^h(i, j) \in R^{N_1 \times 1}\), \(x^v(i, j) \in R^{N_2 \times 1}\) represent the horizontal and vertical state vectors, and \(u(i, j), y(i, j)\) are the input and output, respectively. The coefficient matrices \(A_1(i, j) \in R^{N_1 \times N_1}, A_2(i, j) \in R^{N_1 \times N_2}, A_3(i, j) \in R^{N_2 \times N_1}, A_4(i, j) \in R^{N_2 \times N_2}, B_1(i, j) \in R^{N_1 \times 1}, B_2(i, j) \in R^{N_2 \times 1}, C_1(i, j) \in R^{1 \times N_1}, C_2(i, j) \in R^{1 \times N_2}, D(i, j) \in R\) are functions of \(a_{mn}(i, j)\) and \(b_{mn}(i, j)\) in (1) and periodically shift variant with period \((P, Q)\). The initial conditions are assumed such that \(x^h(i, j) = x^v(i, j) = 0\), \(i < 0\) or \(j < 0\), \(x^h(i, 0) = x^v(i, 0) = 0\), \(i \geq I\) and \(x^h(0, j) = x^v(0, j) = 0, j \geq J\), for some \(I\) and \(J\).

In this paper, the zero-input stability of a GR state-space model (2) is studied so that only the periodic coefficient matrices \(A_1(i, j), A_2(i, j), A_3(i, j),\) and \(A_4(i, j)\) need to be considered. A zero-input 2-D PSV GR model can be written as

\[
\begin{bmatrix}
  x^h(i+1, j) \\
  x^v(i, j+1)
\end{bmatrix} =
\begin{bmatrix}
  A_1(i, j) & A_2(i, j) \\
  A_3(i, j) & A_4(i, j)
\end{bmatrix}
\begin{bmatrix}
  x^h(i, j) \\
  x^v(i, j)
\end{bmatrix}
\]

\[
= A(i, j)
\begin{bmatrix}
  x^h(i, j) \\
  x^v(i, j)
\end{bmatrix}.
\]  

*Electrical & Computer Engineering, Utah State University, Logan, UT 84322-4120. T. Bose’s work was supported by NASA Grant NAG5-10716 and the State of Utah Centers of Excellence program.

**Mathematics & Computer Science, The Citadel, Charleston, SC 29409. M.-Q. Chen’s work was supported in part by a Citadel Foundation Sabbatical Grant and by the Department of Electrical & Computer Engineering, Utah State University.

Electrical & Computer Engineering, St. Cloud State University, St. Cloud, MN 56301.
Note that the GR model can also be written in another form using the second model of Fornasini-Marchesini (FM) [10] as follows. Define
\[
w(i, j) = \begin{bmatrix} x^h(i, j) \\
x^s(i, j) \end{bmatrix}.
\]
Then we can write (3) as
\[
w(i, j) = A^{10}(i-1, j)w(i-1, j) + A^{01}(i-1, j)w(i, j-1)
\] (4)
where
\[
A^{10}(i, j) = \begin{bmatrix} A_1(i, j) & A_2(i, j) \\
0 & 0 \end{bmatrix}
\]
and
\[
A^{01}(i, j) = \begin{bmatrix} 0 & 0 \\
A_3(i, j) & A_4(i, j) \end{bmatrix}
\] (5)
which 0 denotes a zero matrix of appropriate size. The model of (4) is used to obtain several stability conditions.

III. OBSERVATION AND DEFINITIONS

The observations and definitions given in this section are used in deriving the stability conditions in the next section. Observations 1-3 and Definitions 1-5 are established in [5] and therefore are stated below without further explanations.

For vector \( w = \{ w_i \} \), \( |w| = \{|w_i|\} \) and \( |w| \) denotes a vector norm of \( w \). For matrix \( A = \{ A_{ij} \} \), \( |A| = \{|A_{ij}|\} \), \( ||A|| \) denotes an induced matrix norm of \( A \), and \( \rho(A) = \max_i \{ |A_i| : A_i \text{ is the } i\text{th eigenvalue of } A \} \) denotes the spectra radius of \( A \).

For \( P \) and \( Q \) in the model of (4), we let \( P = Sp \) and \( Q = Sq \) where \( p \), \( q \) are positive integers, and \( S \) is the greatest common divisor of \( P \) and \( Q \). Also, let \( R = Spq \) be the least common denominator of \( P \) and \( Q \). Figure 1 shows the indices in the first quadrant of coefficient matrices \( A^{10}(i, j) \) and \( A^{01}(i, j) \) whose periodicity is \( (P, Q) = (4, 6) \).

Observation 1: The state vectors \( w(i, j) \) in every block of size \( P \times Q \) use the same set of coefficient matrices \( A^{10}(i, j) \) and \( A^{01}(i, j) \).

For example, in Figure 1, the state vectors in two \( 4 \times 6 \) blocks use the same set of coefficient matrices.

Observation 2: The state vectors \( w(i, j) \) along each diagonal depend only on a set of \( R \) pairs of coefficient matrices \( A^{10}(i, j) \) and \( A^{01}(i, j) \) when they are related to the state vectors on the previous diagonal.

Observation 3: There are \( S \) different sets of \( R \) pairs of coefficient matrices \( A^{10}(i, j) \) and \( A^{01}(i, j) \).

Definition 1 (Index sets): Index sets for coefficient matrices used by the state vectors \( w(i, j) \) along the diagonal as the first index decreases are defined as follows:
\[
I_r = \{ (r, 0), (r - 1, 1), \ldots, (r - (Q - 1), Q - 1), \}
\]
\[
(r - Q, 0), (r - (Q + 1), 1), \ldots, (r - (2Q - 1), Q - 1), \}
\]
\[
(r - (P - 1)Q, 0), (r - ((P - 1)Q + 1), 1), \ldots, (r - (P - 1)Q + Q - 1, Q - 1), \}
\]
\[
I_r = \{ (j, m) \text{ where } r - nQ - m \equiv j \text{ mod } P, 0 \leq j \leq P - 1 \}.
\]

Label \( R \) ordered elements in the set \( I_r \) by ordered notations: \( i_r(0), \ldots, i_r(R - 1) \) and then
\[
I_r = \{ i_r(0), i_r(1), \ldots, i_r(Q - 1), i_r(Q), \ldots, i_r(R - 1) \}, \quad (6)
\]
where \( r = 0, 1, \ldots, S - 1, m = 0, 1, \ldots, Q - 1 \) and \( n = 0, 1, \ldots, P - 1 \). Note that there are \( R \) different indices in each index set. For example, when \((P, Q) = (4, 6)\), \( S = 2, p = 2, q = 3 \) and \( R = 12 \). The index sets \( I_0 \) and \( I_1 \) are
\[
I_0 = \{ (0, 0), (3, 1), (2, 2), (1, 3), (0, 4), (3, 5), \}
\]
\[
(2, 6), (1, 1), (0, 2), (3, 3), (2, 4), (1, 5) \}
\]
and
\[
I_1 = \{ (1, 0), (0, 1), (3, 2), (2, 3), (1, 4), (0, 5), \}
\]
\[
(3, 0), (2, 1), (1, 2), (0, 3), (3, 4), (2, 5) \}
\].

Definition 2 (Permutation matrices): Let \( T_m \) be an \( m \times m \) cyclic permutation matrix such that \( T_m x_1, x_2, \ldots, x_m \)\( T_m \) = \( x_2, x_3, \ldots, x_m, x_1 \). It is known [15] that \( T_m \) has the following properties.
1. \( T_m T_m^T = I_{m \times m} \) or \( T_m^T = T_m \) where \( I_{m \times m} \) is the \( m \times m \) identity matrix.
2. \( (T_m)^3[x_1, x_2, \ldots, x_{m-1}, x_m]^T = [x_{i+1}, x_{i+2}, \ldots, x_1]^T \).
3. \( (T_m)^m = I_{m \times m} \).

Also, the block cyclic permutation \( T_m = T_m \otimes I_{n \times n} \) where \( \otimes \) is the Kronecker product [12] has the same properties as above described as follows.
1. \( T_m T_m^T = I_{m \times m} \otimes I_{n \times n} \) or \( T_m^T = T_m \).
2. \( (T_m)^2[x_1, x_2, \ldots, x_{m-1}, x_m]^T = [x_{i+1}, x_{i+2}, \ldots, x_{i-1}, x_i]^T \).
3. \( (T_m)^m = I_{m \times m} \otimes I_{n \times n} \).

Definition 3 (Matrix \( G_r^{(i)} \)): For \( r = 0, 1, \ldots, S - 1 \), define matrices \( G_r^{(i)} \) as in (8) and (9) where \( \overline{p} = 1, 2, \ldots, p - 1 \) and \( i_r(n) \in I_r \) as defined in (6). Matrices \( G_r^{(i)} \) have the following properties [5].
1. \( G_r^{(i)} = (T_p)^g G_r (T_p)^g \).
2. \( \prod_{r = S}^{0} G_r^{(i)} = (T_p)^g \prod_{r = S}^{0} G_r^{(i)} (T_p)^g \).
3. \( \prod_{\overline{p} = P - 1}^{0} \prod_{r = S}^{0} G_r^{(i)} = (T_p)^g \prod_{\overline{p} = P - 1}^{0} G_r \).

Definition 4 (2-D system stability): A 2-D state-space system with state vectors \( w(i, j) \) is asymptotically stable if \( \lim_{i+j \to \infty} w(i, j) = 0 \).

Definition 5 (2-D energy function on diagonal): For a 2-D system with state vectors \( w(i, j) \), the energy on the \( k \)th diagonal is defined as
\[
U(k) = \sum_{l=0}^{k} |w(k - l, l)|.
\]

Lemma 1: If \( \lim_{k \to \infty} U(k) = 0 \), then the system is stable.

Proof: If the energy on the progressing diagonals goes to zero, then the state vectors \( w(i, j) \) must also approach zero.
Fig. 1. Indices of coefficient matrices in the first quadrant for a system with period \((P,Q) = (4,6)\)

\[ G_r = G_r^{(0)} = \begin{bmatrix} A^{10}(i_r(0)) & 0 & \cdots & 0 & A^{01}(i_r(R-1)) \\ A^{01}(i_r(0)) & A^{10}(i_r(1)) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^{01}(i_r(R-2)) & A^{10}(i_r(R-1)) \end{bmatrix}. \] (8)

\[ G_r^{(P)} = \begin{bmatrix} A^{10}(i_r(PQ)) & 0 & \cdots & 0 & A^{01}(i_r(PQ-1)) \\ A^{01}(i_r(PQ)) & A^{10}(i_r(PQ+1)) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^{01}(i_r(PQ-2)) & A^{10}(i_r(PQ-1)) \end{bmatrix}. \] (9)

\[ G_r = \begin{bmatrix} A_1(i_r(0)) & A_2(i_r(0)) & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & A_1(i_r(1)) & \cdots & 0 & 0 \\ A_3(i_r(0)) & A_4(i_r(0)) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & A_3(i_r(R-2)) & A_4(i_r(R-1)) & A_3(i_r(R-1)) & A_4(i_r(R-1)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \end{bmatrix}. \] (10)
in any direction in the \((i, j)\) plane. The result follows directly from Definitions 4 and 5.

Next we define an energy function using a vector norm. Definition 6 and Lemma 2 will be used in Theorems 5 and 6.

**Definition 6** (2-D energy function on diagonal): For a 2-D system with state vectors \(w(i, j)\), the energy on the \(k\)th diagonal is defined as

\[
U_{\text{norm}}(k) = \sum_{l=0}^{k} \|w(k-l, l)\|.
\]

**Lemma 2:** If \(\lim_{k \to \infty} U_{\text{norm}}(k) = 0\), then the system is stable.

**Proof:** If the energy using a vector norm on the progressing diagonals goes to zero, then the state vector \(w(i, j)\) must also approach zero in any direction in the \((i, j)\) plane. The result follows directly from Definitions 4 and 6.

The following theorem shows that the results in Lemma 1 and Lemma 2 are equivalent.

**Theorem 1:** Let \(U(k)\) and \(U_{\text{norm}}(k)\) be given in Definition 5 and 6, respectively. Then

\[
\lim_{k \to \infty} U(k) = 0 \text{ if and only if } \lim_{k \to \infty} U_{\text{norm}}(k) = 0.
\]

**Proof:** It is known [13] that vector norms on \(R^n\) or \(C^n\) are equivalent, i.e., for any two vector norms \(\|\cdot\|_\alpha\) and \(\|\cdot\|_\beta\) on \(R^n\) or \(C^n\) there exist finite positive constants \(C_m\) and \(C_M\) such that

\[
C_m \|v\|_\alpha \leq \|v\|_\beta \leq C_M \|v\|_\alpha \text{ for all } v \in R^n \text{ (or } C^n).\]

Assume that \(\lim_{k \to \infty} U(k) = \lim_{k \to \infty} \sum_{i=0}^{k} |w(k-l, l)| = 0\) where \(w(k-l, l) = \{w_i(k-l, l)\}_{i=1}^{N}\). Then \(\lim_{k \to \infty} \sum_{i=0}^{k} |w_i(k-l, l)| = 0\) for \(i = 1, \ldots, N\). Since

\[
\sum_{l=0}^{k} \|w(k-l, l)\| \leq \sum_{k=0}^{\infty} C_M^{(l)} \|w(k-l, l)\|_1
\]

\[
\leq \max_{0 \leq l \leq k} \left\{ C_M^{(l)} \right\} \sum_{l=0}^{k} \|w(k-l, l)\|_1
\]

\[
= \max_{0 \leq l \leq k} \left\{ C_M^{(l)} \right\} \sum_{i=1}^{N} \sum_{l=0}^{k} |w_i(k-l, l)|
\]

\[
= \max_{0 \leq l \leq k} \left\{ C_M^{(l)} \right\} \sum_{i=1}^{N} \sum_{l=0}^{k} |w_i(k-l, l)|
\]

where \(C_M^{(l)}\) are finite positive constants,

\[
\lim_{k \to \infty} U_{\text{norm}}(k) = \lim_{k \to \infty} \sum_{l=0}^{k} \|w(k-l, l)\|
\]

\[
\leq \lim_{k \to \infty} \max_{0 \leq l \leq k} \left\{ C_M^{(l)} \right\} \sum_{i=1}^{N} \sum_{l=0}^{k} |w_i(k-l, l)|
\]

\[
\leq \max_{0 \leq l \leq \infty} \left\{ C_M^{(l)} \right\} \sum_{i=1}^{N} \lim_{k \to \infty} \sum_{l=0}^{k} |w_i(k-l, l)| = 0
\]

Assume that \(\lim_{k \to \infty} U_{\text{norm}}(k) = \lim_{k \to \infty} \sum_{l=0}^{k} \|w(k-l, l)\| = 0\).

Then

\[
\lim_{k \to \infty} \sum_{i=1}^{k} |w_i(k-l, l)| \leq \lim_{k \to \infty} \sum_{i=1}^{k} \sum_{l=0}^{N} |w_i(k-l, l)|
\]

\[
= \lim_{k \to \infty} \sum_{i=1}^{k} \sum_{l=0}^{N} |w_i(k-l, l)|
\]

\[
= \lim_{k \to \infty} \sum_{l=0}^{k} \|w(k-l, l)\|_1
\]

\[
= \max_{0 \leq l \leq \infty} \left\{ C_M^{(l)} \right\} \sum_{i=1}^{N} \sum_{l=0}^{k} |w_i(k-l, l)|
\]

\[
= 0
\]

for some finite positive constants \(C_M^{(l)}\). Since

\[
\lim_{k \to \infty} \sum_{i=1}^{k} |w_i(k-l, l)| = 0 \text{ for } i = 1, \ldots, N,
\]

\[
\lim_{k \to \infty} U(k) = \lim_{k \to \infty} \sum_{i=1}^{k} \|w(k-l, l)\| = 0.
\]

**IV. STABILITY OF 2-D PSV GR-MODEL SYSTEM**

In this section, stability conditions for the stability of 2-D PSV GR models are established using both absolute-value and vector-norm energy functions. The sufficient conditions using the absolute-value energy function are derived in Theorems 2 and 3. Using matrix norms, we derive upper bounds of matrices given in Theorem 3 which lead to four sufficient conditions stated in Corollary 1. Then a necessary condition is given in Theorem 4. Using the vector-norm energy function, sufficient conditions are obtained in Theorems 5 and 6. The results in Theorems 2-4, for state vectors \(w(i, j)\) are similar to the ones for the state variables \(x(i, j)\) established in [5]. Recall that in [5] state variables \(x(i, j)\) of a zero-input 2-D PSV Fornasini-Marchesini state-space system are generated by the recurrence relation

\[
x(i, j) = A(i, j)x(i, j - 1) + B(i, j)x(i - 1, j)
\]

where the coefficients matrices \(A(i, j), B(i, j) \in R^{L \times L}\) are periodically shift variant with period \((P, Q)\) with assumed initial conditions \(x(i, j) = 0, i < 0\) or \(j < 0\), and \(x(i, j) = 0, i \geq I\) or \(j \geq J\). Replacing \(A^{01}(i, j - 1)\) by \(A(i, j)\) and \(A^{10}(i - 1, j)\) by \(B(i, j)\), the recurrence relation for state vectors \(w(i, j)\)

\[
w(i, j) = A^{10}(i - 1, j)w(i - 1, j) + A^{01}(i, j - 1)w(i, j - 1)
\]

is equivalent to the one for state variables \(x(i, j)\) in [5]. So, we state Theorems 2-4 without their proofs but commenting
on the differences.

**Theorem 2:** Consider a zero-input PSV GR model

\[ w(i,j) = A^{10}(i-1,j)w(i-1,j) + A^{01}(i,j-1)w(i,j-1) \]

where \( A^{10}(i,j) \) and \( A^{01}(i,j) \) as defined in (5) are periodic with period \((P,Q)\). The initial conditions are assumed such that \( w(i,j) = 0, \ i < 0 \) or \( j < 0 \) and \( w(0,0) = w(0,j) \) for \( i \geq I, j \geq J \). For \( r = 0, 1, ..., S - 1 \), define

\[
F_r = \max_{0 \leq n \leq R-1, \ i_r(n) \in I_r} \left\{ |A^{10}(i_r(n))| + |A^{01}(i_r(n))| \right\}
= \max_{0 \leq n \leq R-1, \ i_r(n) \in I_r} \{|A(i_r(n))|\}
\tag{12}
\]

where \( I_r \) is the index set defined in (6). If \( \rho \left( \prod_{r=0}^{S-1} F_r \right) < 1 \), then the system is asymptotically stable.

**Proof:** The first equality in (12) follows directly from Theorem 1 in [5] by replacing \( A^{01}(i,j-1) \) and \( A^{10}(i-1,j) \) by \( A(i,j) \) and \( B(i,j) \) as defined in [5], respectively. The second equality

\[ |A^{01}(i_r(n))| + |A^{10}(i_r(n))| = |A(i_r(n))| \]

in (12) holds by the definitions of \( A^{01}(i_r(n)) \) and \( A^{10}(i_r(n)) \).

**Theorem 3:** Consider a zero-input PSV GR model

\[ w(i,j) = A^{10}(i-1,j)w(i-1,j) + A^{01}(i,j-1)w(i,j-1) \]

where \( A^{10}(i,j) \) and \( A^{01}(i,j) \) as defined in (5) are periodic with period \((P,Q)\). The initial conditions are assumed such that \( w(i,j) = 0, \ i < 0 \) or \( j < 0 \) and \( w(0,0) = w(0,j) \) for \( i \geq I, j \geq J \). If

\[
\rho \left( \prod_{r=0}^{S-1} |G_r| \right) < 1 \tag{13}
\]

where \( G_r \) is given by Definition 3, and \( T_p \) is defined in Definition 2 with \( m = p \) and \( n = Q(N_1 + N_2) \), then \( \lim_{i+j \to \infty} w(i,j) = 0 \).

**Proof:** The result follows directly from Theorem 3 in [5] by replacing \( A^{01}(i,j-1) \) and \( A^{10}(i-1,j) \) by \( A(i,j) \) and \( B(i,j) \) as defined in [5], respectively.

The special structure of \( G_r \) enables us to derive some upper bounds for \( \rho \left( \prod_{r=0}^{S-1} |G_r| \right) \). In the following corollary, we use these upper bounds to establish several more restrictive but easy to check sufficient conditions for the stability of 2-D PSV GR models.

**Corollary 1:** Let

\[
\begin{bmatrix}
A_1(i_r(k)) & A_2(i_r(k)) \\
A_3(i_r(k)) & A_4(i_r(k))
\end{bmatrix}
\]

where \( A^{10}(i_r(k)) = \begin{bmatrix} A_1(i_r(k)) & A_2(i_r(k)) \\ 0 & 0 \end{bmatrix} \) and \( i_r(k) \in I_r \). Then \( \rho \left( \prod_{r=0}^{S-1} |G_r| \right) < 1 \) if

1. \[
\max_{0 \leq k \leq R-1} \{ \| A(i_r(k)) \|_2 \} < 1
\]
2. \[
\max_{0 \leq k \leq R-1} \{ \| A(i_r(k)) \|_1 \} < 1;
\]
3. \[
\max_{0 \leq k \leq R-1} \{ \| A(i_r(k)) \|_\infty \} < 1;
\]
4. \[
\max_{0 \leq k \leq R-1} \{ \| A(i_r(k)) \|_\infty \} < 1;
\]

Recall \( G_r \) given in (10), let \( G_r \) be a matrix \( \begin{bmatrix} g_{i,j}^{(r)} \end{bmatrix} \in R^{(N_1+N_2) \times (N_1+N_2)} \), \( A_1(i_r(k)) = \begin{bmatrix} a_{i,j}^{(1)}(i_r(k)) \end{bmatrix} \in R^{N_1 \times N_1} \), \( A_2(i_r(k)) = \begin{bmatrix} a_{i,j}^{(2)}(i_r(k)) \end{bmatrix} \in R^{N_1 \times N_2} \), \( A_3(i_r(k)) = \begin{bmatrix} a_{i,j}^{(3)}(i_r(k)) \end{bmatrix} \in R^{N_2 \times N_1} \), and \( A_4(i_r(k)) = \begin{bmatrix} a_{i,j}^{(4)}(i_r(k)) \end{bmatrix} \in R^{N_2 \times N_2} \).

1. Observe that \( G_r \) is a block diagonal matrix where for \( k = 0, 1, ..., R-1 \), and

\[
\begin{bmatrix}
E_0 & & \\
& \ddots & \\
& & E_R
\end{bmatrix}
\]

(14)

Let \( \lambda_i(C) \) be the \( i \)th eigenvalue of \( C \). Then

\[
\| G_r \|_2 = \max_{0 \leq k \leq R-1} \left\{ \lambda_i(\| G_r^T \| G_r) \right\}
= \max_{0 \leq k \leq R-1} \left\{ \| A(i_r(k)) \|_2 \right\}
\]

and

\[
\max_{0 \leq k \leq R-1} \{ \| A(i_r(k)) \|_2 \} < 1.
\]
implies $\rho \left( T_p^T \prod_{r=S-1}^0 |G_r| \right) \leq \prod_{r=S-1}^0 \| G_r \|_2 < 1$.

(2) Observe that

$$\|G_r\|_1 = \max_{1 \leq j \leq R(N_1+N_2)} \sum_{i=1}^{R(N_1+N_2)} |g_{i,j}^{(r)}|$$

implies

$$\|G_r\|_1 = \max_{1 \leq j \leq N_1} \left\{ \max_{1 \leq j \leq N_2} \left\{ \begin{array}{l}
\sum_{i=1}^{N_1} |a_{i,j}^{(1)}(i_r(k))| + \sum_{i=1}^{N_2} |a_{i,j}^{(2)}(i_r(k))| \\
\sum_{i=1}^{N_1} |a_{i,j}^{(3)}(i_r(k))| + \sum_{i=1}^{N_2} |a_{i,j}^{(4)}(i_r(k))|
\end{array} \right\} \right\},$$

where

$$\|G_r\|_\infty = \max_{0 \leq k \leq R-1} \left\{ \max_{0 \leq k \leq R-1} \left\{ \begin{array}{l}
\sum_{i=1}^{N_1} |a_{i,j}^{(1)}(i_r(k))| + \sum_{i=1}^{N_2} |a_{i,j}^{(2)}(i_r(k))| \\
\sum_{i=1}^{N_1} |a_{i,j}^{(3)}(i_r(k))| + \sum_{i=1}^{N_2} |a_{i,j}^{(4)}(i_r(k))|
\end{array} \right\} \right\}.$$
By taking a vector norm on both sides, the triangular inequality
where $G_r$ is given by Definition 3, and $\overline{T}_p$ is defined in
Definition 2 with $m = p$ and $n = Q(N_1 + N_2)$.

**Proof:** The result follows directly from Theorem 2 in [5]
by replacing $A^{01}(i, j - 1)$ and $A^{10}(i - 1, j)$ by $A(i, j)$ and $B(i, j)$ as defined in [5], respectively.

As shown in [5], the condition of Theorem 2 is more restrictive than that of Theorem 3. However, the required computation in Theorem 2 is less than the one in Theorem 3 since the size of $F_r$ is $(N_1 + N_2) \times (N_1 + N_2)$ whereas the size of $|G_r|$ is $Q(N_1 + N_2) \times Q(N_1 + N_2)$.

The next two sufficient conditions for stability are derived
by using the vector-norm energy function. The advantage is
that the sizes of matrices in the conditions are greatly reduced
so that the required computation is significantly less than those
in Theorems 2 and 3.

**Theorem 5:** Consider a zero-input PSV GR model

\[
A^{10}(i - 1, j)w(i - 1, j) + A^{01}(i, j - 1)w(i, j - 1)
\]

where $A^{10}(i, j)$ and $A^{01}(i, j)$ as defined in (5) are periodic
with period $(P, Q)$. The initial conditions are assumed such
that $w(i, j) = 0$, $i < 0$ or $j < 0$ and $w(0, 0) = w(0, j)$ for
$i \geq I$, $j \geq J$. For $r = 0, 1, ..., S - 1$, define

\[
\bar{F}_r = \max_{1 \leq n \leq R - 1, \sum_{i \in i_r} \|A^{10}(i_r(n))\| + \|A^{01}(i_r(n))\|}
\]

where $I_r$ is the index set defined in (index). If \( \prod_{r = S - 1}^{0} \bar{F}_r < 1 \), then the system is asymptotically stable.

**Proof:** Using the system description and the initial
conditions, the state variables $w(n_1, n_2)$ on the $(k + 1)$th
diagonal can be written as

\[
\begin{align*}
w(k + 1, 1) &= A^{10}(k, 0)w(k, 0), \\
w(k, 1) &= A^{01}(k, 0)w(k, 0), \quad w(k, 2) = A^{10}(k - 1, 1)w(k - 1, 1) + A^{01}(k, 0)w(k, 0), \\
w(k - 1, 2) &= A^{01}(k - 1, 1)w(k - 1, 1), \\
& \vdots \\
w(1, k) &= A^{10}(0, k)w(0, k), \\
w(0, k + 1) &= A^{01}(0, 1)w(1, k - 1), \\
w(0, k + 1) &= A^{01}(0, k)w(0, k).
\end{align*}
\]

By taking a vector norm on both sides, the triangular inequality
of a vector norm and the property of an induced matrix norm
give us the following inequalities:

\[
\|w(k + 1, 1)\| \leq \|A^{10}(k, 0)\|\|w(k, 0)\|,
\]

\[
\|w(k, 1)\| \leq \|A^{01}(k, 0)\|\|w(k, 0)\|.
\]

Adding the terms on both sides of inequalities and combining
like terms together on the right-hand side, the following
inequality holds:

\[
\sum_{l=0}^{k+1} (\|w(k + 1 - l, l)\|) \leq \sum_{l=0}^{k} (\|A^{10}(k - l, l)\| + \|A^{01}(k - l, l)\|) \|w(k - l, l)\|. 
\]

Applying the upper bound on the right-hand side, we have

\[
\|w(n(k + 1))\| \leq \left( \max_{0 \leq l \leq k} \{\|A^{10}(k - l, l)\| + \|A^{01}(k - l, l)\| \} \right) U_{\text{norm}}(k)
\]

where

\[
U_{\text{norm}}(k) = \sum_{l=0}^{k} \|w(k - l, l)\|.
\]

Using the definition for $\bar{F}_r$ (16), we obtain

\[
U_{\text{norm}}(k + 1) \leq \bar{F}_k U_{\text{norm}}(k).
\]

By this inequality, we have

\[
U_{\text{norm}}(k) \leq \bar{F}_{k-1} \bar{F}_{k-2} \cdots \bar{F}_{k-(nS+T)} U_{\text{norm}}(k - (nS + T)).
\]

Since $\bar{F}_k$ is periodic with period $S$, by using recursion we
have for $k = r \mod S$,

\[
U_{\text{norm}}(k) \leq \left( \prod_{l=1}^{k} \bar{F}_l \right) \left( \prod_{l=S-1}^{0} \bar{F}_l \right)^{n-1} \left( \prod_{l=S-1}^{(r-T) \mod S} \bar{F}_l \right) U_{\text{norm}}(k - (nS + T)),
\]

where $T \leq r$ and $\sum_{l=1}^{0} \bar{F}_l = 1$. Hence, if \( \left( \prod_{l=S-1}^{0} \bar{F}_l \right) \leq 1 \), then \( \lim_{k \to \infty} U_{\text{norm}}(k) = 0 \).

Note that because for $\alpha = 1, 2, \text{ or } \infty$,

\[
\|A(k)\| = \|A^{10}(k) + A^{01}(k)\| \leq \|A^{10}(k)\| + \|A^{01}(k)\|.
\]

\[
\max_{r = S-1}^{0} \|G_r\| \leq \max_{r = S-1}^{0} \left\{ \|A^{(r)}(k)\| \right\}.
\]
\[ \leq \prod_{r=S-1}^{0} \max_{0 \leq k \leq R-1} \{ \| A^{10}(k) \|_\alpha + \| A^{01}(k) \|_\alpha \} \]
\[ \leq \prod_{r=S-1}^{0} \tilde{F}_r. \]

Hence, \( \left( \prod_{i=S-1}^{0} \tilde{F}_i \right) < 1 \) implies \( \left( \prod_{r=S-1}^{0} \| G_r \|_\alpha \right) < 1 \) for \( \alpha = 1, 2, \) or \( \infty. \)

**Theorem 6:** Consider a zero-input PSV GR model
\[ w(i, j) = A^{10}(i-1,j) w(i-1, j) + A^{01}(i, j-1) w(i, j-1) \]
where \( A^{10}(i, j) \) and \( A^{01}(i, j) \) as defined in (5) are periodic with period \((P, Q).\) The initial conditions are assumed such that \( w(i, j) = 0, i < 0 \) or \( j < 0 \) and \( w(i, 0) = w(0, j) \) for \( i \geq I, j \geq J. \) If
\[ \rho \left( \prod_{p=1}^{0} \tilde{G}_r \right) < 1 \]
(17)
where \( \tilde{G}_r \) is defined in (18) and \( \prod_{p=1}^{0} \) is defined in Definition 2 with \( m = p \) and \( n = Q, \) then \( \lim_{i+j \to \infty} w(i, j) = 0. \)

**Proof:** Using the system description and the initial conditions and letting \( k = l \text{ mod } P, \) the state variables \( w(n_1, n_2) \) on the \((k+1)\)th diagonal can be written as
\[ w(k+1, 0) = A^{10}(l, 0) w(k, 0), \]
\[ w(k, 1) = \left( A^{10}((l-1)_{mod \ P}, 0) w(k, 1) + A^{01}(l, 0) w(k, 0), \right) \]
\[ w(k-1, 2) = \left( A^{10}((l-2)_{mod \ P}, 1) w(k, 2) + A^{01}((l-1)_{mod \ P}, 1) w(k, 1) \right), \]
\[ \vdots \]
\[ w(k+1 - R, R) = \left( A^{10}(k - R, R) w(k - R, R) + A^{01}(k + 1 - R, R - 1) w(k + 1 - R, R - 1) \right) \]
\[ = \left( A^{10}(l, 0) w(k, - R, R) + A^{01}((l+1)_{mod \ P}, Q - 1) w(k + 1 - R, R - 1) \right) \]
\[ w(k+1 - (R+1), R+1) = \left( A^{10}(k - (R+1), R+1) w(k - (R+1), R+1) + A^{01}(k + 1 - (R+1), R) w(k + 1 - (R+1), R) \right) \]
\[ = \left( A^{10}((l-1)_{mod \ P}, 1) w(k - (R+1), R) + A^{01}(l, 0) w(k + 1 - (R+1), R) \right), \]
(21)
\[ \vdots \]
\[ w(0, k+1) = A^{01}(0, k_{mod Q}) w(0, k). \]

Notice that every \( R \)th state vector on the diagonal, i.e. (20) and (21), employs the same coefficient matrices. Therefore, we first take a vector norm and then combine all state vectors that use the same coefficient matrices i.e.
\[ \left[ \| w(k+1, 0) \| + \| w(k+1 - R, R) \| \right] + \| w(k+1 - (R-1), R-1) \| + \| w(k+1 - (2R-1), 2R-1) \| + \| w(k+1 - (3R-1), 3R-1) \| + \ldots \]

By the triangular inequality, we have \( R \) different inequalities as the second index increases from 0 to \( R - 1.\)

For \( k+1 = MR + N, \) where \( M \) is a positive integer and \( 0 \leq N \leq R - 1, \) define
\[ t_i(k) = \begin{cases} M & 0 \leq i \leq N - 1 \\ M - 1 & N - 1 < i \leq R - 1 \end{cases} \]
\[ t_i(k+1) = \begin{cases} M & 0 \leq i \leq N - 1 \\ M - 1 & N - 1 < i \leq R - 1 \end{cases}. \]
The above is the total number of state vectors on the \((k+1)\)th diagonal which use the same coefficient matrices \( A^{10}(i, j) \) and \( A^{01}(i, j) \) when they are represented in terms of the state variables on the previous diagonal. The \( R \) different inequalities can be written in matrix form as follows. Letting
\[ W_{\text{norm}}(k+1) = \left[ \begin{array}{c} t_i(k+1) \sum_{m=0}^{N} ||w(k+1 - Rm, Rm)|| \\ t_i(k+1) \sum_{m=0}^{N} ||w(k - Rm, Rm + 1)|| \\ \vdots \end{array} \right], \]
the inequalities can be expressed as
\[ W_{\text{norm}}(k+1) \leq \tilde{G}_l W_{\text{norm}}(k), \quad k = l \text{ mod } P, \]
(22)
where \( \tilde{G}_l \) is defined in (19). Since \( \tilde{G}_l \) repeats with period \( P, \) as the first index increases, we can write using a recursion on (22) as:
\[ W_{\text{norm}}(k) \leq \left( \prod_{l=L-1}^{0} \tilde{G}_l \right) \left( \prod_{l=P-1}^{0} \tilde{G}_l \right)^{n-1} \]
\[ \left( L-T \right)^{mod \ P} \prod_{l=P-1}^{0} \tilde{G}_l \right) W_{\text{norm}}(k - (nP + T)) \]
(23)
for \( k = L \text{ mod } P, \) \( T \leq L \) and \( \prod_{l=L-1}^{0} \tilde{G}_l = I \) (identity matrix). Then \( \lim_{k \to \infty} W_{\text{norm}}(k) = 0 \) if \( \rho \left( \prod_{r=S-1}^{0} \tilde{G}_l \right) < 1. \)

The proof of the fact that \( \rho \left( \prod_{r=S-1}^{0} \tilde{G}_l \right) < 1 \) if and only if
\[ \rho \left( \prod_{r=S-1}^{0} \tilde{G}_l \right) < 1 \] is given in Lemma 3 in Appendix. ■

As shown in the note after the proof of Theorem 5, conditions in (1), (2) and (3) in Corollary 1 are less restrictive than that of Theorem 5. Lemma 4 (Appendix) shows that the
condition in Theorem 5 is more restrictive than the one in Theorem 6. As pointed out earlier, sufficient conditions (2), (3) and (4) in Corollary 1 do not require the computation of a spectral radius of a matrix, and sizes of matrices involved in the computation in Theorems 5 and 6 are greatly reduced since the size of matrix $F_r$ in Theorem 2 is $(N_1 + N_2) \times (N_1 + N_2)$ whereas $\hat{F}_r$ in Theorem 5 is a scalar and the size of matrix $|G_r|$ in Theorem 3 is $Q(N_1 + N_2) \times Q(N_1 + N_2)$ whereas the size of matrix $\hat{G}_r$ in Theorem 6 is $Q \times Q$. Thus, the computations required to check the conditions in Corollary 1, Theorem 5 and 6 using 1-norm, inf-norm and Frobenius-norm are significantly less than those in Theorems 2, and 3. However, systems which satisfy the condition of Theorem 6 always satisfy the condition of Theorem 3, i.e.,

$$
\rho \left( T_p^T \prod_{r=S-1}^{0} |G_r| \right) \leq \rho \left( T_p^T \prod_{r=S-1}^{0} \hat{G}_r \right).
$$

The proof of this inequality is given in Lemma 8.

V. ALGORITHM AND EXAMPLES

In the previous section, five sufficient conditions (counting the necessary condition in Corollary 1 as one) and one necessary condition for stability of a 2-D PSV GR model are established in Theorems 2, 3, 5 and 6, Corollary 1, and Theorem 4, respectively. It is also discussed that sufficient conditions (1), (2) and (3) in Corollary 1 are less restrictive than the one in Theorem 5, and sufficient conditions in Theorems 2, 5 and 6 are more restrictive than the one in Theorem 3, and the sufficient condition of Theorem 5 is more restrictive than the one of Theorem 6. The advantage of applying Theorem 2, Corollary 1, Theorem 5 or Theorem 6 is that the corresponding sufficient condition is relatively easy to be checked because the sizes of matrices given in these sufficient conditions are much smaller than the one in Theorem 3. The sizes of $F_r$, $|G_r|$, $F_r$, $\hat{F}_r$ and $\hat{G}_r$ in Theorems 2, 3, 4, 5 and 6, and $A(i_r(k))$ in Corollary 1, respectively, are given in Table 1, along with their sufficient and necessary conditions. Note that sufficient conditions in Corollary 1, Theorem 5 and Theorem 6 using 1-norm, inf-norm or Frobenius-norm are much easier to compute than the one using 2-norm, which requires computing eigenvalues of $A^T A$ of a given matrix $A$ and has the same computational complexity as the one for computing the spectral radius of $A$. For a fair comparison, we do not use 2-norm in our examples.

Hence, to check the stability of a given 2-D PSV system in GR model, it is natural to start with conditions (2) or (3) in Corollary 1, and then condition (4) in Corollary 1 or the condition in Theorem 5 using Frobenius, and then the conditions in: Theorem 6, Theorem 2, Theorem 4 and Theorem 3 in this order. The system is stable if one of the sufficient conditions is satisfied, is not stable if the necessary condition fails, and no conclusion is reached if all sufficient conditions fail but the necessary condition holds. The algorithm is presented by a flowchart in Figure 2.

Six examples are given below to illustrate the stability conditions and to show how the algorithm works. For simplicity, we consider zero-input PSV GR models with period $(2, 2)$, that is, $P = 2$, $Q = 2$, $p = 1$, $q = 1$, $S = 2$ and $R = 2$. By Definition 1, index sets are:

$$
I_0 = \{i_0(0), i_0(1)\} = \{(0, 0), (1, 1)\},
$$

$$
I_1 = \{i_1(0), i_1(1)\} = \{(1, 0), (0, 1)\}.
$$

We let $N_1 = N_2 = 2$ so sizes of $A^{10}(i, j)$ and $A^{01}(i, j)$ are $4 \times 4$ matrices. For each example, we list coefficient matrices

$$
A(i_j(k)) = \begin{bmatrix}
A_1(i_j(k)) & A_2(i_j(k)) \\
A_3(i_j(k)) & A_4(i_j(k))
\end{bmatrix}
$$

for $j=0,1$ and $k=0,1$ in MatLab format and summarize the numerical results in Table 2.

**Example 1:** $A((0,0)) = \begin{bmatrix}
.13 & .05 & .05 & .02 \\
.2 & .1 & .0 & .1 \\
.3 & .1 & .2 & .1 \\
.2 & .3 & .1 & .1
\end{bmatrix}$,

$$
A((1,1)) = \begin{bmatrix}
.2 & .3 & .1 & .2 \\
.1 & .2 & .3 & .1 \\
.2 & .1 & .3 & .1 \\
.2 & .1 & .1 & .1
\end{bmatrix},
$$

$$
A((0,1)) = \begin{bmatrix}
.2 & .1 & .3 & .1 \\
.2 & .1 & .1 & .1 \\
.1 & .1 & .2 & .1 \\
.0 & .1 & .1 & .2
\end{bmatrix},
$$

$$
A((1,0)) = \begin{bmatrix}
.3 & .3 & .3 & .3 \\
.3 & .2 & .1 & .4 \\
.2 & .1 & .2 & .1 \\
.0 & .1 & .1 & .1
\end{bmatrix},
$$

**Example 2:** $A((0,0)) = \begin{bmatrix}
.01 & .01 & .01 & .01 \\
.01 & .01 & .01 & .01 \\
.01 & .01 & .01 & .01 \\
.01 & .01 & .01 & .01
\end{bmatrix}$,

$$
A((1,1)) = \begin{bmatrix}
.01 & .02 & .01 & .01 \\
.01 & .02 & .01 & .01 \\
.01 & .02 & .01 & .01 \\
.01 & .02 & .01 & .01
\end{bmatrix},
$$

$$
A((0,1)) = \begin{bmatrix}
.03 & .04 & .03 & .01 \\
.03 & .04 & .03 & .01 \\
.03 & .04 & .03 & .01 \\
.03 & .04 & .03 & .01
\end{bmatrix},
$$

$$
A((1,0)) = \begin{bmatrix}
.01 & .01 & .01 & .01 \\
.01 & .01 & .01 & .01 \\
.01 & .01 & .01 & .01 \\
.01 & .01 & .01 & .01
\end{bmatrix}.
<table>
<thead>
<tr>
<th>Thm</th>
<th>Cor</th>
<th>Condition</th>
<th>matrix or scalar</th>
<th>size</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td>$\rho \left( \prod_{r=S-1}^{0} F_r \right) &lt; 1$</td>
<td>$F_r$</td>
<td>$(N_1 + N_2) \times (N_1 + N_2)$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$\rho \left( T_p^T \prod_{r=S-1}^{0}</td>
<td>G_r</td>
<td>\right) &lt; 1$</td>
</tr>
</tbody>
</table>
|     | 1   | $\begin{cases} 
(1) \prod_{r=S-1}^{0} \max_k \{ \| A_i(k) \|_2 \} < 1 \\
(2) \prod_{r=S-1}^{0} \max_k \{ \| A_i(k) \|_1 \} < 1 \\
(3) \prod_{r=S-1}^{0} \max_k \{ \| A_i(k) \|_\infty \} < 1 \\
(4) \prod_{r=S-1}^{0} \sqrt{\sum_{i=0}^{R-1} \| A_i(k) \|_F^2} < 1 
\end{cases}$ | $A_i(k)$ | $(N_1 + N_2) \times (N_1 + N_2)$ |
| 4   |     | $\rho \left( T_p^T \prod_{r=S-1}^{0} G_r \right) < 1$ | $G_r$ | $Q(N_1 + N_2) \times Q(N_1 + N_2)$ |
| 5   |     | $\prod_{r=S-1}^{0} \bar{F}_r < 1$ | $\bar{F}_r$ | $1 \times 1$ |
| 6   |     | $\rho \left( T_p^T \prod_{r=S-1}^{0} \bar{G}_r \right) < 1$ | $\bar{G}_r$ | $Q \times Q$ |

Table 1: Sizes of matrices involved in stability conditions

<table>
<thead>
<tr>
<th>Ex</th>
<th>Cor</th>
<th>1 suff cond.</th>
<th>Thm 2 suff cond.</th>
<th>Thm 5 suff cond.</th>
<th>Thm 6 suff cond.</th>
<th>Thm 4 nece cond.</th>
<th>Thm 3 suff cond.</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>(3) 0.63</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td>2</td>
<td>(2)</td>
<td>1.10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td>1.40</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td>0.924</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td>3</td>
<td>(2)</td>
<td>1.08</td>
<td>(4) 1.270</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td>1.17</td>
<td></td>
<td>(2) 0.821</td>
<td></td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td>1.14</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td>4</td>
<td>(2)</td>
<td>1.17</td>
<td>(4) 1.858</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td>1.10</td>
<td></td>
<td>(2) 1.23</td>
<td>(3) 2.799</td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td>1.512</td>
<td></td>
<td>(4) 1.492</td>
<td></td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td>5</td>
<td>(2)</td>
<td>5.60</td>
<td>(4) 6.729</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Unstable</td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td>4.40</td>
<td></td>
<td>(2) 6.043</td>
<td>(3) 11.91</td>
<td></td>
<td></td>
<td>Unstable</td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td>6.245</td>
<td></td>
<td>(4) 6.195</td>
<td></td>
<td></td>
<td></td>
<td>Unstable</td>
</tr>
<tr>
<td>6</td>
<td>(2)</td>
<td>2.85</td>
<td>(4) 2.889</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td>2.40</td>
<td></td>
<td>(2) 2.149</td>
<td>(3) 4.696</td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td>2.51</td>
<td></td>
<td>(4) 2.253</td>
<td></td>
<td></td>
<td></td>
<td>Stable</td>
</tr>
</tbody>
</table>

Table 2: Numerical results

Example 3: $A((0,0)) =\begin{bmatrix} 0.1 & 0.1 & 0 & 0.4 & -0.4 & 0.1 & 0.1 & 0;0.1 \end{bmatrix}$

Example 4: $A((0,0)) =\begin{bmatrix} 0.1 & 0.2 & 0.4 & -0.1 & 0.2 & -0.2 & -0.1 & 0.2;0.2 & -0.3 & 0.4 & 0.2 \end{bmatrix}$

Example 5: $A((0,0)) =\begin{bmatrix} 0.5 & 0.1 & 0.7 & -0.1 & -0.2 & 0.2 & 0.8 & 2.0;0.5 & 0.4 & 0.7 & 0.6 & 0.1 & 0.3 & 0.6 \end{bmatrix}$
is therefore stable.

For example 5, we test the sufficient conditions of the Corollary, Theorem 5, and Theorem 6 with the same matrix norms used in Example 4, and then Theorem 2 and none is satisfied. Then we test the necessary condition of Theorem 4, which is not satisfied and the system is therefore unstable.

In example 6, we go down the flow chart testing each condition in sequence until we reach Theorem 3. The sufficient condition of Theorem 3 is satisfied and hence this system is stable. In this final example, we had to go through the entire algorithm to come to a conclusion about its stability.

VI. CONCLUSION

The stability of 2-D PSV filters represented by the Givone-Roesser state-space model is studied. Several sufficient conditions and one necessary condition are established for the stability. Sufficient conditions given in Theorems 2 and 3, and Corollary 1, and one necessary condition are obtained using the absolute value energy function, and sufficient conditions given in Theorems 5 and 6 are obtained using the vector norm energy function. The sufficient conditions given in Corollary 1, Theorem 2, Theorem 5, and Theorem 6 are more restrictive than the one given in Theorem 3 and the sufficient condition given in Theorem 5 is more restrictive than the one given in Theorem 6. However, the conditions given in Corollary 1 and Theorem 5 using 1-norm, inf-norm and Frobenius-norm and condition in Theorem 2 are simpler in computation and are easy to apply. The necessary condition given in Theorem 4 can also be used to quickly check for instability of a given system. An algorithm for checking stability is proposed and six examples are given to illustrate the results.

VII. APPENDIX

Lemma 3: Consider matrix \( \tilde{G}_r \) defined in (19), and matrix \( \hat{G}_r \) defined in (18). Then \( \rho \left( \prod_{i=0}^{p-1} \tilde{G}_r \right) < 1 \) if and only if \( \rho \left( \prod_{r=S-1}^{p} \hat{G}_r \right) < 1 \).

Proof: Observe that the matrices \( \tilde{G}_r, \hat{G}_{S+r}, \cdots, \hat{G}_{(p-1)S+r} \) use the same coefficient matrices with indices in \( I_r \) by shifting \( Q \) indices each other. So \( \prod_{i=0}^{p-1} \tilde{G}_r \) can be simplified further. Let \( l = pS + r \), \( \tilde{G}_r = \hat{G}_r^{(p)} \), where \( \hat{G}_r^{(p)} \) are defined in (9). Then \( \prod_{i=0}^{p-1} \hat{G}_r = \prod_{i=0}^{p-1} \left( \prod_{r=S-1}^{p} \hat{G}_r^{(p)} \right) \) by the properties given in Property 3 in Definition 3. Hence \( \rho \left( \prod_{i=0}^{p-1} \tilde{G}_r \right) < 1 \) if and only if \( \rho \left( \prod_{r=S-1}^{p} \hat{G}_r \right) < 1 \). ■

The following two results are given in [5] and are used in the proof of Lemma 6.

Lemma 4: (5), Lemma 1)Let \( A = [A_{i,j}], B = [B_{i,j}] \) be in \( R_{\min \times \min} \), where \( A_{i,j}, B_{i,j} \in R_{\min \times \min} \), and let \( C = AB = [C_{i,j}] \) where \( C_{i,j} \in R_{\max \times \max} \). If \( \sum_{i=0}^{m-1} A_{i,j} = E_A \) and \( \sum_{i=0}^{m-1} B_{i,j} = E_B \), then \( \sum_{i=0}^{m-1} C_{i,j} = E_A E_B \), for \( j = 0, 1, \cdots, m-1 \).
Lemma 5: ([5], Lemma 2) Let \( A = [A_{i,j}] \) in \( R^{m \times n} \times R^{x \times n} \) where \( A_{i,j} \in R^{n \times n} \). If \( A \geq 0 \) and \( \sum_{i=0}^{m-1} A_{i,j} = E \) for \( j = 0, 1, \ldots, m-1 \), then \( \rho(A) = \rho(E) \).

Lemma 6: Given \( \tilde{F}_r \) defined in (16) and \( \tilde{G}_r \) defined in (18), then
\[
\rho \left( \prod_{r = S-1}^{0} \tilde{G}_r \right) \leq \rho \left( \prod_{r = S-1}^{0} \tilde{F}_r \right).
\]

Proof: Observe that \( T_p \tilde{G}_{S-1} = \tilde{G}_{S-1} \) and the column sum of \( T_p^{-1} \tilde{G}_{S-1} \) are the same as the one of \( \tilde{G}_{S-1} \). Observe also that the elements of the column sum of \( G_r \) which are \( \| A_{10}(i_r(0)) \| + \| A_{20}(i_r(0)) \|, \ldots, \| A_{10}(i_r(R-1)) \| + \| A_{20}(i_r(R-1)) \| \) are always less than or equal to
\[
\tilde{G}_r = \max_{0 \leq n \leq R-1, \quad i_r(n) \in I_r} \{ \| A_{10}(i_r(n)) \| + \| A_{20}(i_r(n)) \| \}.
\]
Define matrices \( \tilde{G}_r \) for \( r = 0, 1, \ldots, S-1 \) such that all elements of the column sum of \( \tilde{G}_r \) are equal to \( \tilde{F}_r \) by adding the positive values to those elements whose values are not equal to \( \tilde{F}_r \). Clearly, \( \tilde{G}_r \leq \tilde{G}_r \) for \( r = 0, 1, \ldots, S-1 \). Then
\[
\rho \left( \prod_{r = S-1}^{0} \tilde{G}_r \right) = \rho \left( \prod_{r = S-1}^{0} \tilde{F}_r \right) \leq \rho \left( \prod_{r = S-1}^{0} \tilde{G}_r \right).
\]
Since each \( \tilde{G}_r \) has a constant column sum \( \tilde{F}_r \), the column sum of \( \tilde{G}_r \) is the same as the one of \( \tilde{G}_r \) by Lemma 4. By Lemma 5, \( \rho \left( \prod_{r = S-1}^{0} \tilde{G}_r \right) = \rho \left( \prod_{r = S-1}^{0} \tilde{F}_r \right). \) Therefore
\[
\rho \left( \prod_{r = S-1}^{0} \tilde{F}_r \right) \leq \rho \left( \prod_{r = S-1}^{0} \tilde{G}_r \right) = \rho \left( \prod_{r = S-1}^{0} \tilde{F}_r \right).
\]
Thus, if \( \rho \left( \prod_{r = S-1}^{0} \tilde{F}_r \right) < 1 \), then \( \rho \left( \prod_{r = S-1}^{0} \tilde{G}_r \right) < 1 \) which implies that the condition in Theorem 5 is more restrictive than the one in Theorem 6.

The following result is given in [14] and is used in the proof of Lemma 8.

Lemma 7: ([14], Theorem 2.2): Let \( C(r) = [C_{i,j}(r)] \) be block \( m \times m \) matrices for \( r = 1, 2 \), where \( C_{i,j}(r) \) are nonnegative \( N_i \times N_j \) matrices for \( i,j = 1, \ldots, m \). Define \( B(r) = \| [C_{i,j}(r)] \| \) for \( r = 1, 2 \), and \( \| \cdot \| \) is a consistent matrix norm. Then \( \rho \left( C(2) C(1) \right) \leq \rho \left( B(2) B(1) \right) \).

Lemma 8: Consider \( \tilde{G}_r \) and \( \tilde{G}_r \) given in (8) and (18), respectively.
\[
\rho \left( \prod_{r = S-1}^{0} \tilde{G}_r \right) < 1 \text{ if and only if } \rho \left( \prod_{r = S-1}^{0} \tilde{F}_r \right) < 1.
\]

Proof: Define \( \tilde{G}_l \) as in (24) for \( l = 0, 1, \ldots, P-1 \),
\[
\rho \left( \prod_{r = S-1}^{0} \tilde{G}_r \right) < 1 \text{ if and only if } \rho \left( \prod_{r = S-1}^{0} \tilde{F}_r \right) < 1.
\]
From Lemma 3, we know
\[
\rho \left( \prod_{r = P}^{0} \tilde{G}_r \right) < 1 \text{ if and only if } \rho \left( \prod_{r = P}^{0} \tilde{F}_r \right) < 1.
\]
Therefore, it suffices to show that \( \rho \left( \prod_{l = P-1}^{0} \tilde{G}_l \right) < 1 \) then
\[
\rho \left( \prod_{l = P-1}^{0} \tilde{G}_l \right) < 1 \text{ if and only if } \rho \left( \prod_{l = P-1}^{0} \tilde{F}_l \right) < 1.
\]
We first show \( \rho \left( \prod_{l = P-1}^{0} \tilde{G}_l \right) \leq \rho \left( \prod_{l = P-1}^{0} \tilde{F}_l \right) \) by Mathematical Induction. It follows directly from Lemma 7 that
\[
\rho \left( \prod_{l = P-1}^{0} \tilde{G}_l \right) \leq \rho \left( \prod_{l = P-1}^{0} \tilde{F}_l \right).
\]
Therefore assume that the inequality \( \rho \left( \prod_{l = P-1}^{0} \tilde{G}_l \right) \leq \rho \left( \prod_{l = P-1}^{0} \tilde{F}_l \right) \) for some \( r \).

Because \( \rho \left( \prod_{l = P-1}^{0} \tilde{G}_l \right) \leq \rho \left( \prod_{l = P-1}^{0} \tilde{F}_l \right) \),
\[
\rho \left( \prod_{l = P-1}^{0} \tilde{G}_l \right) < 1 \text{ implies } \rho \left( \prod_{l = P-1}^{0} \tilde{F}_l \right) < 1.
\]

Acknowledgement We would like to thank the anonymous reviewers for their constructive comments, which led to a significant improvement of the paper.

REFERENCES

\[
\hat{G}_l = \begin{bmatrix}
A^{10}(l, 0) & 0 & \cdots & 0 \\
A^{01}(l, 0) & A^{10}((l - 1) \mod P, 1) & \cdots & 0 \\
0 & A^{01}((l - 1) \mod P, 1) & \ddots & \cdots \\
0 & 0 & \cdots & A^{10}((l + 1) \mod P, Q - 1)
\end{bmatrix}.
\]

(24)


Fig. 1. Indices of coefficient matrices in the first quadrant for a system with period \((P,Q) = (4,6)\).

Fig. 2. Flowchart - Stability testing algorithm

Manuscript Received:
Affiliation of authors:
T. Bose, Electrical and Computer Engineering, Utah State University, Logan, UT 84322-4120. T. Bose’s work was supported by NASA Grant NAG5-10716 and the State of Utah Centers of Excellence program.
M.-Q. Chen, Mathematics and Computer Science, The Citadel, Charleston, SC 29409. M.-Q. Chen’s work was supported in part by a Citadel Foundation Sabbatical Grant and by the Department of Electrical and Computer Engineering, Utah State University
R. Thamvichai, Electrical and Computer Engineering, St. Cloud State University, St. Cloud, MN 56301