A Summary of Mei Chen’s Sabbatical Proposal:

This sabbatical proposal consists of **three research projects** and **one teaching project**. Detailed descriptions of these four projects are given below.

**Objective I.**

The first objective is to complete a joint in progress research project on “The optimal Successive Over-Relaxation Method for a class of shifted companion matrices,” a joint work with Dr. Xiezhang Li, professor and acting chair of Mathematical Sciences at Georgia Southern University.

Let \( A \) be an \( n \times n \) non-negative shifted companion matrix whose Jacobi matrix \( J \) is of the form:

\[
J = \begin{bmatrix}
0 & b_2 & \cdots & b_{n-1} & b_n \\
1 & 0 & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & 0
\end{bmatrix},
\]

where \( b_i \geq 0, \ i = 2, \ldots, n-1, \ b_n > 0 \) and \( \sum_{i=2}^{n} b_i < 1. \)

Let \( J = L + U \) where \( L \) and \( U \) are the strictly lower and upper triangular matrices of \( J \), respectively. When the Successive Over-Relaxation (SOR) method is used to solve the system of linear equations: \( Ax = b \), the iterative matrix \( L_\omega \) with an over-relaxation parameter \( \omega \) is given by

\[
L_\omega = (I - \omega L)^{-1}((1 - \omega)I + \omega U).
\]

The iterative matrix \( L_{\omega_b} \) of the optimal (the best) SOR method for the matrix \( A \) has the property that the spectral radius \( \rho(L_{\omega_b}) \) of \( L_{\omega_b} \) is minimized over all \( L_\omega \) for \( 0 \leq \omega \leq 2 \). In 1954, Young derived the over-relaxation parameter \( \omega_b \) for the optimal SOR method for a matrix whose Jacobi matrix \( J \) is 2-cyclic and in 1959, Varga derived \( \omega_b \) for the optimal SOR method for a matrix whose Jacobi matrix \( J \) is \( p \)-cyclic where \( p \geq 2 \). The over-relaxation parameter \( \omega_b \) for the optimal SOR method is not known for a general matrix. The matrix \( J \) defined above is 2-cyclic if \( b_2 > 0 \) and \( b_k = 0 \) for \( k \geq 3 \) and is \( n \)-cyclic if \( b_n \neq 0 \) and \( b_k = 0 \) for \( k = 2, \ldots, n-1 \). The goal of this project is to find \( \omega_b \) for \( A \) when \( J \) is not a 2-cyclic and is not a \( n \)-cyclic matrix.

We have characterized the spectral radius \( \rho(L_\omega) \) of \( L_\omega \) for \( 0 \leq \omega \leq 2 \). Based on the characterizations, we have derived formulas for the overrelaxation parameter \( \omega_b \), and a corresponding positive real eigenvalue \( \lambda(\omega_b) \) of \( L_{\omega_b} \). We have proved that the
spectral radii $\rho(L_{\omega})$ of $L_{\omega}$ are greater than $\lambda(\omega_b)$ for all $0 \leq \omega < \omega_b$. In order to show $L_{\omega_b}$ is the optimal SOR iterative matrix for $0 \leq \omega \leq 2$, we need to complete the proofs of the following two results.

1) $\lambda(\omega_b)$ is the spectral radius $\rho(L_{\omega_b})$ of $L_{\omega_b}$.

2) The spectral radii $\rho(L_{\omega})$ of $L_{\omega}$ are also greater than the spectral radius $\rho(L_{\omega_b})$ of $L_{\omega_b}$ for $\omega_b < \omega \leq 2$.

**Importance.**

1) Systems of linear equations $Ax = b$ where $A$ is an non-negative shifted companion matrix arise in mathematical biological population models. The optimal SOR method solves efficiently such systems.

2) The behaviors of the spectral radius $\rho(L_{\omega})$ of a SOR iterative matrix $L_{\omega}$ for general matrices are not known. The over-relaxation parameter $\omega_b$ for the optimal SOR method is not known for a general matrix. Our study of the optimal SOR method for a class of shifted companion matrices will be an important step for the further study the optimal SOR method for general matrices

**Objective II.**

The second objective is to complete a joint in progress research project on “Spectral properties of a row-stochastic Leslie matrix,” a joint work with Dr. Xiezhang Li, professor and acting chair of mathematics at Georgia Southern University.

A Leslie matrix is a square and nonnegative matrix in which all elements are zero except the ones in the first row and the ones right below the main diagonal. A row-stochastic Leslie matrix $A$ is a Leslie matrix in which the sum of the first row is 1 and elements right below the main diagonal are 1. Row-stochastic Leslie matrices arise in mathematical models for population growth. The eigenvalues of a row-stochastic Leslie matrix are important in describing the limiting behavior of the corresponding population model. The second largest eigenvalue of $A$ in module, denoted by $\lambda_2$, determines the rate of convergence of the system $X(k + 1) = AX(k)$ to a steady-state solution. If $\lambda_2$ is known, then the rate of convergence can be identified and therefore can be improved. It is not known any efficient algorithm to find $\lambda_2$ for a given $A$.

In 1993, Kirkland studied the eigenvalue distribution of a row-stochastic Leslie matrix with a near-periodic fecundity pattern which arises from biology models for some species. Let $\lambda_j = r_j e^{i\theta_j}$ be the $j$th largest eigenvalue of $A$ in module. Kirkland developed an algorithm that computes certain number of intervals such that each of them contains at least one $\theta_j$. Using Kirkland's algorithm, we have further studied the distribution of eigenvalues of $A$, where $\theta_j$ are in these intervals and learned from a huge number of numerical experiments that one specific interval contains the argument $\theta_2$ of $\lambda_2$. We are now in the process to prove this numerical testing result is theoretically
true. After that, we plan to develop an algorithm that will first estimate an interval \( I \) that contains \( \theta_2 \) for a row-stochastic Leslie matrix \( A \) and then compute \( \theta_2 \) using the Power Method with a starting point \( \lambda_0 = r_0 \ e^{i \theta_0} \) for \( \theta_0 \) in the interval \( I \).

**Importance.**

Finding efficiently a steady-state solution of the system \( X(k+1) = AX(k) \) where \( A \) is a row-stochastic Leslie matrix is important to many population applications. Knowing the exact value or a good estimate of \( \lambda_2 \), the second largest eigenvalue of \( A \) in module, will allow one to improve the rate of convergence of the system \( X(k+1) = AX(k) \) to a steady-state solution. An algorithm will be developed to find efficiently a close estimate of \( \lambda_2 \).

**Objective III.**

The third objective is to complete a new project on “Conjugate Gradient algorithm for the non-negative matrix factorizations and applications in signal and image processing,” a joint work with Dr. Tamal Bose, professor and chair of Electrical and Computer Engineering at the Utah State University.

Non-negative matrix factorization (NMF) is a mathematical process that finds non-negative matrix factors \( W \) and \( H \) for a given non-negative matrix \( V \) such that \( V = WH \). It is a useful decomposition for multivariate data. In 1997, a gradient type algorithm was suggested to approximate \( W \) and \( H \) by minimizing the cost function

\[
 f(W,H) = \| V - WH \|_F^2 \quad \text{where} \quad \| A \|_F^2 = \sum_{i,j} a_{ij}^2 \quad \text{for} \quad A = [a_{ij}],
\]

which measures the square of the Euclidean distance between \( V \) and \( WH \). This algorithm has been successfully employed for NMF in many areas of applications, such as image processing, text information retrieval, and machine learning. The advantage of this algorithm is its simplicity. The disadvantage is that it is not always stable and it has a slow rate of convergence. Recently, a singular value decomposition (SVD) type technique is used for symmetric non-negative matrix factorization when the non-negative matrix \( V \) is also symmetric. Such a symmetric non-negative matrix occurs in video and media summarization technology. This algorithm is robust and stable but its computational complexity is very high. A question naturally arises “Can we develop a stable algorithm for NMF that solves a problem faster than the gradient type algorithm with less computational complexity as one of SVD?” It is known in the field of scientific computing that the Conjugate Gradient (CG) algorithm is stable and converges much faster than a gradient type algorithm for solving a system of linear equations. Dr. Bose and I have written a journal paper on the CG algorithm for solving least squares problems arising in the design of bilinear filters and I have written a journal paper on the CG algorithm for solving systems of linear equations with many right hand sides. From our experience, we believe that the CG algorithm will outperform the gradient type algorithm
and in some cases it may outperform a SVD type algorithm. For this project, our tasks are the following.

1) Develop an algorithm for NMF using the CG algorithm. Test the algorithm and determine how fast the algorithm converges both numerically and theoretically.

2) Explore applications for NMF in signal and image processing.

3) Write journal papers to report our research results.

Importance.

A new stable and faster algorithm will be developed for non-negative matrix factorization. The algorithm is expected to be stable, outperform the existing gradient type algorithm and require less computation than the singular value decomposition type algorithm. Applications of NMF in signal and image processing will be studied.

Objective IV.

The fourth objective is to create a set of Java applets for teaching courses: Calculus I, II and III, Linear Algebra, Applied Mathematics I and II and Numerical Methods I and II.

Java applets are programs written in Java that can be embedded into web pages to enable sophisticated interactions and simulations. They can be used to illustrate difficult concepts with a sequence of changing diagrams, which a user can control. Java applets have become powerful teaching and learning tools. I have been using Java applets through internet for classroom demonstrations in teaching calculus. However, I cannot always find a Java applet for the topic I need, especially, for topics in a high level mathematics course. Sometimes, the source server is down when I need to use a Java applet in teaching. Hence, I want to develop a set of Java applets for courses that I teach.

Funded by the Citadel Foundation Faculty Development Grant, I will be attending a mini-course on "Java Applets in Teaching Mathematics" at the MAA-AMS Joint Winter Conference held in Phoenix, Arizona, January 7-19. I am planning to use the Java programming techniques and the Visual Development Environment and a Math Toolkit presented at the mini-course to develop mathematical Applets in the spring, 2004, for topics of limit, definition of a derivative, definition of a definite integral, and the Mean Value Theorem for Calculus I that I will be teaching in the spring. I will use these applets for class demonstration and will also add them into my online lecture notes for Calculus I so students can involve with these mathematical activities outside classroom.

Through my sabbatical, I will develop a set of Java applets for topics in courses: Calculus II and III, Linear Algebra, Engineering Mathematics I and II, and Numerical Methods I and II. I have been teaching these courses regularly. A list of topics that I am planning to create a Java applet is given in the detail proposal.

Importance.

A set of Java applets for topics in courses Calculus, Linear Algebra, Engineering Mathematics and Numerical Methods will be created. These Java applets will enable me to improve my teaching and students learning.
I plan to spend the whole academic year of 2004-2005 plus the whole summer of 2005 to complete all four projects. A detailed plan is given in the following table.

<table>
<thead>
<tr>
<th>Time period</th>
<th>Project</th>
<th>Activity</th>
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| Jul – Aug, 2004 | 1, 4    | 1. Complete the proof of that $\omega_b$ is the optimal SOR parameter and start on the proof of that the spectral radii of $L(\omega)$ are greater than the spectral radius of $L(\omega_b)$ for $\omega_b < \omega < 2$.  
2. Develop Java Applets for Calculus II and III. |
| Sept – Oct, 2004 | 1, 4    | 1. Complete the proof of that the spectral radii of $L(\omega)$ are greater than the spectral radius of $L(\omega_b)$ for $\omega_b < \omega < 2$ and start the writing of the paper to report all obtained results on the optimal SOR Method for a class of shifted companion matrices.  
2. Develop Java Applets for Linear Algebra and Numerical Method I. |
| Nov – Dec, 2004 | 1, 2    | 1. Complete the writing of the paper to report all obtained results on the optimal SOR Method for a class of shifted companion matrices.  
2. Study the spectral property of a row-stochastic Leslie matrix with a near-periodic fecundity pattern. Find the interval where the second largest eigenvalue in module is located. |
| Jan – Feb, 2005 | 2, 4    | 1. Study the spectral property of a row-stochastic Leslie matrix. Find the interval where the second largest eigenvalue in module is located.  
2. Design an algorithm to find the second largest eigenvalue in module for a row-stochastic Leslie matrix.  
3. Develop Java Applets for Numerical Method II. |
| Mar - Apr, 2005 | 2, 3    | 1. Write a paper to report obtained results on the spectral property of a row-stochastic Leslie matrix.  
2. Develop an algorithm for NMF using the CG algorithm. |
| May – Aug, 2005 | 3, 4    | 1. Implement the CG type algorithm for NMF in MatLab and test the algorithm. Compare the algorithm with the gradient and SVD type algorithms.  
2. Write a paper to report the results.  
3. Develop Java Applets for Engineering Mathematics I and II. |