Review Notes for the Calculus I/Precalculus Placement Test - Fall, 2006

Part 4 -

1. **Solution of a polynomial equation: factoring, and grouping**
   If a polynomial can be factored as a product of linear or quadratic polynomials, for example,
   $$P = 4(2x-1)(x+3)^2(x^2+x-1)$$
   then the solutions of the equation $P = 0$ are solutions of the corresponding linear equations or quadratic equations. The solutions of $P = 0$ for P given above are:
   $$\begin{cases} 2x-1=0, \quad x=\frac{1}{2} \\ x+3=0, \quad x=-3 \\ x^2+x-1=0, \quad x=\dfrac{-1\pm\sqrt{5}}{2} \end{cases}, \text{ so, solutions are } \begin{cases} x_1 = \frac{1}{2} \\ x_2 = -3 \\ x_3 = \dfrac{-1+\sqrt{5}}{2} \\ x_4 = \dfrac{-1-\sqrt{5}}{2} \end{cases}.$$
   Expressing a given polynomial as a product of polynomials with less degrees is called **factoring**.
   a. **Identifying common factors:**
      Identify all common factors in a given polynomial and then factor them out. The polynomial becomes a product of the common factors and a resulting polynomial. For example, the polynomial
      $$2x^2(x-1)(x+1) - 8x(x-1)(2x+1)$$
      has common factors 2, x and $x-1$. It can be written as
      $$\begin{aligned} 2x^2(x-1)(x+1) - 8x(x-1)(2x+1) &= 2x(x-1)(x(x+1) - 4(2x+1)) \\ &= 2x(x-1)(x^2+x-8x-4) \\ &= 2x(x-1)(x^2-7x-4) \end{aligned}$$
      The solutions of the equation $2x^2(x-1)(x+1) - 8x(x-1)(2x+1) = 0$ are solutions of the equation
      $$2x(x-1)(x^2-7x-4) = 0$$
      which can be solved as follows.
      $$\begin{cases} x = 0 \\ x-1=0, \ x=1 \\ x^2-7x-4=0, \ x = \dfrac{7\pm\sqrt{7^2-4(1)(-4)}}{2(1)} = \dfrac{7\pm\sqrt{65}}{2} \end{cases}.$$
      Solutions are:
      $$x_1 = 0, \quad x_2 = 1, \quad x_3 = \frac{7+\sqrt{65}}{2}, \quad x_4 = \frac{7-\sqrt{65}}{2}.$$

   b. **Factoring the difference of two squares, perfect square, difference of two cubes or sum of two cubes.**
      Recall:

Difference of two squares	$x^2 - a^2 = (x-a)(x+a)$
Perfect squares	$x^2 \pm 2ax + a^2 = (x \pm a)^2$
Difference of two cubes	$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$
Sum of two cubes	$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$

Example *Solve the equation by factoring.*

i. $16x^4 - 81 = 0$

$16x^4 - 81$ *is of the form* $a^2 - b^2$ *with* $a = 4x^2$ *and* $b = 9$. *So,*

$$16x^4 - 81 = (4x^2)^2 - 9^2 = (4x^2 - 9)(4x^2 + 9)$$

The factor $4x^2 - 9$ *is again of the form* $a^2 - b^2$ *with* $a = 2x$ *and* $b = 3$. *Therefore,*

$$16x^4 - 81 = (4x^2 - 9)(4x^2 + 9) = (2x-3)(2x+3)(4x^2+9).$$

Note that $4x^2 + 9 > 0$ *for any* x. *So, solutions of* $16x^4 - 81 = 0$ *are*

$$\begin{cases} 2x - 3 = 0 \\ 2x + 3 = 0 \end{cases} \Rightarrow \begin{array}{l} x_1 = \dfrac{3}{2} \\ x_2 = -\dfrac{3}{2} \end{array}.$$

ii. $x^3 + 27 = 0$

$x^3 + 27$ *is of the form* $a^3 + b^3$ *with* $a = x$ *and* $b = 3$. *So,*

$$x^3 + 27 = (x+3)(x^2 - 3x + 9).$$

The discriminant of the equation $x^2 - 3x + 9 = 0$ *is* $D = (-3)^2 - 4(1)(9) = -27 < 0$. *So, the equation* $x^2 - 3x + 9 = 0$ *has no real solution. The solution of the equation* $x^3 + 27 = 0$ *is* $x = -3$.

iii. $x^6 - 64 = 0$

$x^6 - 64$ *is of the form* $a^3 - b^3$ *with* $a = x^2$ *and* $b = 4$. *So,*

$$x^6 - 64 = (x^2)^3 - 4^3 = (x^2 - 4)(x^4 + 4x^2 + 16) = (x-2)(x+2)(x^4 + 4x^2 + 16).$$

Since $x^4 + 4x^2 + 16 \geq 16$ *for any* x, *the solutions of the equation* $x^6 - 64 = 0$ *are*

$$x_1 = 2, \quad \text{and } x_2 = -2.$$

iv. $x^4 - 4x^2 + 4 = 0$

$x^4 - 4x^2 + 4$ *is a perfect square* $(x^2 - 4)^2$ *which can be further factored as*

$$(x^2 - 4)^2 = ((x-2)(x+2))^2 = (x-2)^2(x+2)^2$$

So the solutions of the equation $x^4 - 4x^2 + 4 = 0$ *are* $x_1 = 2$ *and* $x_2 = -2$.

c. **Factoring by grouping:**

Sometimes a common factor does not occur in every term of the polynomial, but in each of several groups of terms. The common factor can be factored out if these groups are identified. This technique is called factoring by grouping. For example, terms of the polynomial

$$x^3 - 4x^2 - 2x + 8$$

do not have a common factor. However, the first two terms x^3 and $-4x^2$ have a common factor x^2

and the last two terms $-2x$ and 8 have a common factor -2. So, the polynomial can be written as a sum of two groups:

$$x^2(x-4)-2(x-4).$$

These two groups have a common factor $(x-4)$. Factoring $(x-4)$ out, we have

$$(x-4)(x^2-2) = (x-2)\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right).$$

We also use the factors for the difference of two squares. Solutions of the equation

$$(x-4)(x^2-2) = 0$$

are $x_1 = 2$, $x_2 = \sqrt{2}$ and $x_3 = -\sqrt{2}$.

2. **Definition of a function**

   a. **Relations**

   A relation is a **correspondence** between 2 sets. Let x be an element of one set and y be an element of the other set. If a relation exists between x and y, then we say that x **corresponds** to y or that y **depends** on x. We write the relation of x and y as **an ordered pair** $(x,\ y)$.

   b. **Functions**

   Let A and B be **two non-empty sets**. **A function** f from A to B is **a relation** that associates with each element of A **exactly one element** of B. The set A is called **the domain** of the function f. For each element x in A the corresponding element $y = f(x)$ is called the value of the function at x or the image of x. The set of all images of the elements of the domain A is called **the range** of the function f. Here are some examples of functions.

$$f(x) = x, \qquad f(x) = x^2, \qquad f(x) = x^3, \qquad f(x) = \sqrt{x}, \qquad f(x) = \frac{1}{x}$$

   i. **Domain of a function**

   The domain of a function f is **the largest set** D which contains all real numbers x at which $f(x)$ is defined.

Example *Find the domain of the function f.*

A. $f(x) = x^3 + 5x$

$f(x)$ is a polynomial, and its domain is the set of all real numbers, or $D = (-\infty,\ \infty)$.

B. $f(x) = \dfrac{3x}{x^2-4}$

$f(x)$ is not defined whenever **its denominator is zero**:

$$x^2-4 = (x-2)(x+2) = 0,\ x = 2,\ x = -2.$$

So the domain of f is the set containing all real numbers except $x = 2$ and $x = -2$, or

$$D = \{x:\ x \neq 2 \text{ and } x \neq -2\} \text{ or } D = (-\infty,\ -2) \cup (-2,\ 2) \cup (2,\ \infty).$$

C. $f(t) = \sqrt{4-3t}$

$f(t)$ is defined if $4-3t \geq 0$. That is $3t \leq 4$, or $t \leq \frac{4}{3}$. So the domain of $f(t)$ is the set containing all real numbers which are less than or equal to $\frac{4}{3}$, or

$$D = \left\{t:\ t \leq \frac{4}{3}\right\} \text{ or } D = (-\infty,\ \tfrac{4}{3}]\ .$$

   ii. **Operations on Functions**

   A. **Scalar multiplication, sum, difference, product and quotient:**

$$cf(x), \quad f(x)+g(x), \quad f(x)-g(x), \quad f(x)g(x), \quad \frac{f(x)}{g(x)}$$

B. **Composition of functions:**

$$(f \circ g)(x) = f(g(x)) \qquad (g \circ f)(x) = g(f(x))$$

Example *Let* $f(x) = 2x^2 - 3x$. *Evaluate the following.*

A. $f(3)$

$$f(3) = 2(3)^2 = 18$$

B. $f(2x) + f(3)$

$$f(2x) + f(3) = 2(2x)^2 - 3(2x) = 8x^2 - 6x$$

C. $f(-x)$

$$f(-x) = 2(-x)^2 - 3(-x) = 2x^2 + 3x$$

D. $-f(x)$

$$-f(x) = -(2x^2 - 3x) = -2x^2 + 3x$$

E. $f(2x+3)$

$$f(2x+3) = 2(2x+3)^2 - 3(2x+3) = 8x^2 + 18x + 9$$

F. $f(3h) + 1$

$$f(3h) + 1 = 2(3h)^2 - 3(3h) + 1 = 18h^2 - 9h + 1$$

Example *Let* $f(x) = x^2 - 1$, *and* $g(x) = \sqrt{x^2+1}$. *Evaluate the following.*

A. $f(1) + g(0)$

$$f(1) + g(0) = 0 + 1 = 1$$

B. $f(x) - g(x)$

$$f(x) - g(x) = x^2 - 1 - \sqrt{x^2+1}$$

C. $f(2)g(0)$

$$f(2)g(0) = (4-1)(1) = 3$$

D. $\dfrac{f(x)}{g(x)}$

$$\frac{f(x)}{g(x)} = \frac{x^2-1}{\sqrt{x^2+1}}$$

E. $(f \circ g)(1)$

$$(f \circ g)(1) = f(g(1)) = f\left(\sqrt{2}\right) = \left(\sqrt{2}\right)^2 - 1 = 2 - 1 = 1$$

F. $(g \circ f)(x)$

$$(g \circ f)(x) = g(f(x)) = g(x^2 - 1) = \sqrt{(x^2 - 1)^2 + 1} = \sqrt{x^4 - 2x^2 + 1 + 1}$$
$$= \sqrt{x^4 - 2x^2 + 2}$$