Review Notes for the Calculus I/Precalculus Placement Test - Fall, 2006

Part 9 -

1. **Degree and radian angle measures**
   a. **Relationship between degrees and radians**

1 **degree** $= \frac{\pi}{180}$ **radian**	1 **radian** $= \frac{180}{\pi}$ **degree**

Example *Convert each angle in radians to degrees.*

i. $\theta = \frac{\pi}{6}$ *radians* $= \frac{\pi}{6}\left(\frac{180}{\pi}\right) = 30^\circ$
ii. $\theta = 2$ *radians* $= 2\left(\frac{180}{\pi}\right) = \frac{360}{\pi}^\circ \approx 114.59^\circ$
iii. $\theta = -\frac{3\pi}{4}$ *radians* $= -\frac{3\pi}{4}\left(\frac{180}{\pi}\right) = -135^\circ$
iv. $\theta = \frac{7\pi}{3}$ *radians* $= \frac{7\pi}{3}\left(\frac{180}{\pi}\right) = 420^\circ$

Example *Convert each angle in degrees to radians.*

i. $\theta = 45^\circ = 45\left(\frac{\pi}{180}\right) = \frac{\pi}{4}$ *radians*
ii. $\theta = -30^\circ = -30\left(\frac{\pi}{180}\right) = -\frac{\pi}{6}$ *radians*
iii. $\theta = 150^\circ = 150\left(\frac{\pi}{180}\right) = \frac{5\pi}{6}$ *radians*
iv. $\theta = 200^\circ = 200\left(\frac{\pi}{180}\right) = \frac{10\pi}{9}$ *radians*

   b. **Arc length**
   For a circle of radius r, a central angle of θ **in radians** subtends an arc whose length s is

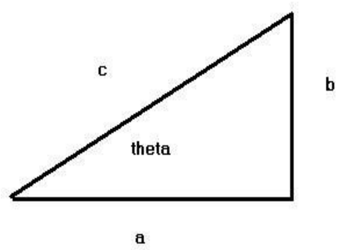
$$s = r\theta.$$

Example *Find the length of the arc of a circle of radius 2 meters subtended by a central angle of* 15°.

Solution *To compute s, we need to convert θ to radians first.*

$$s = (2)\left(15\left(\frac{\pi}{180}\right)\right) = \frac{1}{6}\pi \approx 0.52\ m.$$

2. **Right triangle trigonometry:** $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$
   Consider a **right triangle**. Let θ be one of two **acute angles**. Let the length of the side that is **opposite** θ be b and the side that is **adjacent** to θ be a and the the **hypotenuse** be c. See below.



By the Pythagorean Theorem,
$c^2 = a^2 + b^2$.

Six trigonometric functions of acute angles are defined as

$\sin\theta = \frac{b}{c}$	$\cos\theta = \frac{a}{c}$
$\tan\theta = \frac{b}{a} = \frac{\sin\theta}{\cos\theta}$	$\cot\theta = \frac{a}{b} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$
$\csc\theta = \frac{c}{b} = \frac{1}{\sin\theta}$	$\sec\theta = \frac{c}{a} = \frac{1}{\cos\theta}$

Since a, b and c are positive, values of these trigonometric functions of θ are positive when θ is acute.

Example *Find the value of each 6 trigonometric functions of the angle θ if we know $c = 4$ and $a = 3$.*

Solution *Find b first: $b^2 = c^2 - a^2 = 16 - 9 = 7$, $b = \sqrt{7}$. Then*

$$\sin\theta = \frac{\sqrt{7}}{4}, \quad \cos\theta = \frac{3}{4}, \quad \tan\theta = \frac{\sqrt{7}}{3}, \quad \cot\theta = \frac{3}{\sqrt{7}}$$

$$\csc\theta = \frac{4}{\sqrt{7}}, \quad \sec\theta = \frac{4}{3}.$$

Example *Let θ be an acute angle of a right triangle. Suppose we know $\sin\theta = \frac{1}{3}$ and $\cos\theta = \frac{\sqrt{8}}{3}$. Find the value of each of the four remaining trigonometric functions of θ.*

Solution *By the definitions:*

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{8}}, \quad \cot\theta = \frac{1}{\tan\theta} = \sqrt{8}, \quad \csc\theta = \frac{1}{\sin\theta} = 3, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{3}{\sqrt{8}}$$

3. **Identities:** $\sin^2\theta + \cos^2\theta = 1$

Since

$$\sin^2\theta + \cos^2\theta = \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = \frac{b^2 + a^2}{c^2} = \frac{c^2}{c^2} = 1$$

and

$$\sec^2\theta - \tan^2\theta = \left(\frac{c}{a}\right)^2 - \left(\frac{b}{a}\right)^2 = \frac{c^2 - b^2}{a^2} = \frac{a^2}{a^2} = 1$$

we have the identities

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{and} \quad \sec^2\theta - \tan^2\theta = 1.$$

Example *Let θ be an acute angle of a right triangle. Suppose we know $\sin\theta = \frac{1}{4}$. Find the value of* *each of the five remaining trigonometric functions of θ.*

Solution *By the identity:* $\sin^2\theta + \cos^2\theta = 1$

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{1}{4}\right)^2 = 1 - \frac{1}{16} = \frac{15}{16}, \quad \cos\theta = \frac{\sqrt{15}}{4} \text{ (since } \theta \text{ is acute)}$$

Then

$$\tan\theta = \frac{1}{\sqrt{15}}, \quad \cot\theta = \sqrt{15}, \quad \sin\theta = 4, \quad \cos\theta = \frac{4}{\sqrt{15}}.$$

Example *Let θ be an acute angle of a right triangle. Suppose we know $\tan\theta = \frac{1}{2}$. Find the value of each of the five remaining trigonometric functions of θ.*

Solution *Since* $\tan\theta = \frac{b}{a} = \frac{1}{2}$, *let* $b = 1$, $a = 2$. *Then* $c = \sqrt{a^2 + b^2} = \sqrt{5}$ *and*

$$\sin\theta = \frac{1}{\sqrt{5}}, \quad \cos\theta = \frac{2}{\sqrt{5}}, \quad \cot\theta = 2, \quad \csc\theta = \sqrt{5}, \quad \sec\theta = \frac{\sqrt{5}}{2}.$$

4. **Values of trigonometric functions at special angles**

Let $\theta = \frac{\pi}{4} = 45^\circ$, and $a = b = 1$. Then $c = \sqrt{1^2 + 1^2} = \sqrt{2}$. Values of six trigonometric functions are

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
$\tan\frac{\pi}{4} = \frac{1}{1} = 1$	$\cot\frac{\pi}{4} = \frac{1}{1} = 1$
$\csc\frac{\pi}{4} = \sqrt{2}$	$\sec\frac{\pi}{4} = \sqrt{2}$

Let $\theta = \frac{\pi}{6} = 30^\circ$, $b = 1$, and $c = 2$. Then $a = \sqrt{2^2 - 1^2} = \sqrt{3}$. Values of six trigonometric functions are

$\sin\frac{\pi}{6} = \frac{1}{2}$	$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	$\cot\frac{\pi}{6} = \frac{\sqrt{3}}{1} = \sqrt{3}$
$\csc\frac{\pi}{6} = \frac{2}{1} = 2$	$\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

Let $\theta = \frac{\pi}{3} = 60^\circ$, $a = 1$, and $c = 2$. Then $b = \sqrt{2^2 - 1^2} = \sqrt{3}$. Values of six trigonometric functions are

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$	$\cos\frac{\pi}{3} = \frac{1}{2}$
$\tan\frac{\pi}{3} = \frac{\sqrt{3}}{1} = \sqrt{3}$	$\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\csc\frac{\pi}{3} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	$\sec\frac{\pi}{3} = \frac{2}{1} = 2$

When $\theta = 0$, the right triangle is a horizontal line segment. So, $b = 0$ and $a = c$. Let $a = c = 1$. Then values of six trigonometric functions are

$\sin 0 = \frac{0}{1} = 0$	$\cos 0 = \frac{1}{1} = 1$
$\tan 0 = \frac{0}{1} = 0$	$\cot 0 =$ does not exist
$\csc 0 =$ does not exist	$\sec 0 = \frac{1}{1} = 1$

Similarly, when $\theta = \frac{\pi}{2}$, the right triangle is a vertical line segment. So, $a = 0$ and $b = c$. Let $b = c = 1$. Then values of six trigonometric functions are

$\sin\frac{\pi}{2} = \frac{1}{1} = 1$	$\cos\frac{\pi}{2} = \frac{0}{1} = 0$
$\tan\frac{\pi}{2} =$ does not exist	$\cot\frac{\pi}{2} = \frac{0}{1} = 0$
$\csc\frac{\pi}{2} = \frac{1}{1} = 1$	$\sec\frac{\pi}{2} =$ does not exist

In summary,

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\csc\theta$	$\sec\theta$
$0(0^\circ)$	0	1	0	DNE	DNE	1
$\frac{\pi}{6}(30^\circ)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	2	$\frac{2}{\sqrt{3}}$
$\frac{\pi}{4}(45^\circ)$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}(60^\circ)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	2
$\frac{\pi}{2}(90^\circ)$	1	0	DNE	0	1	DNE

Example *Find the exact value of each expression.*

a. $\sin\frac{\pi}{4}\tan\frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)(1) = \frac{\sqrt{2}}{2}$

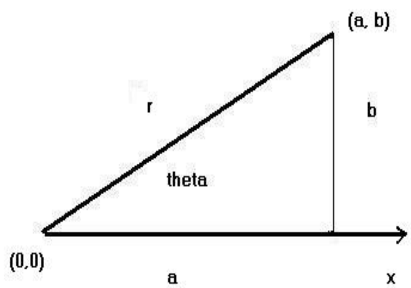
b. $\sec\frac{\pi}{3}\cot\frac{\pi}{3} = (2)\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}$

c. $\sin 30^\circ + \cos 30^\circ = \left(\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right) = \frac{1+\sqrt{3}}{2}$

d. $\csc\frac{\pi}{2} + \sec 0 = 1 + 1 = 2$

5. **Values of trigonometric functions at general angles**

To extend the definitions of the trigonometric functions to **general angles**, we consider an angle in a rectangular coordinate system. An angle is in standard position if its vertex is at the origin and its initial side is along the positive x –axis. Let θ be an angle in standard position and let $(a,\ b)$ denote the coordinates of a point, except the origin $(0,\ 0)$, on the terminal side of θ. Let $r = \sqrt{a^2 + b^2}$.



Then

$\sin\theta = \frac{b}{r}$	$\cos\theta = \frac{a}{r}$
$\tan\theta = \frac{b}{a}$	$\cot\theta = \frac{a}{b}$
$\csc\theta = \frac{r}{b}$	$\sec\theta = \frac{r}{a}$

Both a and b can positive and negative. The signs of the trigonometric functions are:

Quadrant where θ lies	$\sin\theta,\ \csc\theta$	$\cos\theta,\ \sec\theta$	$\tan\theta,\ \cot\theta$
I $(a > 0,\ b > 0)$	+	+	+
II $(a < 0,\ b > 0)$	+	–	–
III $(a < 0,\ b < 0)$	–	–	+
IV $(a > 0,\ b < 0)$	–	+	–

Example *Find the exact value of each of the six trigonometric functions of a positive angle θ if $(4,-1)$ is a point on its terminal side.*

Solution *First find r :* $a = 4,\ b = -1,\quad r = \sqrt{4^2 + (-1)^2} = \sqrt{17}$. *The point $(4,-1)$ is in the 4th quadrant. By definition:*

$$\sin\theta = -\frac{1}{\sqrt{17}},\qquad \cos\theta = \frac{4}{\sqrt{17}},\qquad \tan\theta = -\frac{1}{4},\qquad \cot\theta = -\frac{4}{1} = -4$$

$$\csc\theta = -\sqrt{17},\qquad \sec\theta = \frac{\sqrt{17}}{4}$$

Example *Find the exact value of each of the six trigonometric functions when $\theta = \pi$ and $\theta = \frac{3\pi}{2}$.*

Solution *Let $\theta = \pi$. Then the point on the terminal side is $(-a,\ 0)$ where $a > 0$ and* $r = \sqrt{(-a)^2 + 0^2} = a$. *So,*

$$\sin\pi = \frac{0}{a} = 0, \qquad \cos\pi = \frac{-a}{a} = -1,$$
$$\tan\pi = \frac{0}{-1} = 0, \qquad \cot\pi = \textit{does not exist}$$
$$\csc\pi = \textit{does not exist}, \qquad \sec\pi = \frac{1}{-1} = -1.$$

Let $\theta = \frac{3\pi}{2}$*. Then the point on the terminal side is* $(0,\ -b)$ *where* $b > 0$ *and* $r = \sqrt{0^2 + (-b)^2} = b$*. So,*

$$\sin\frac{3\pi}{2} = \frac{-b}{b} = -1, \qquad \cos\frac{3\pi}{2} = \frac{0}{b} = 0,$$
$$\tan\frac{3\pi}{2} = \textit{does not exist}, \qquad \cot\frac{3\pi}{2} = \frac{0}{-b} = 0$$
$$\csc\frac{3\pi}{2} = \frac{1}{-1} = -1, \qquad \sec\frac{3\pi}{2} = \textit{does not exist}.$$

Example *Name the quadrant in which the angle* θ *lies.*

a. $\sin\theta > 0$, *and* $\cos\theta < 0$
   θ *is in the second quadrant.*
b. $\cos\theta > 0$, *and* $\cot\theta < 0$
   θ *is in the fourth quadrant.*