2.3 The Fixed-Point Algorithm

1. Mean Value Theorem:

Theorem **Rolle’s Theorem**: Suppose that \( f \) is continuous on \([a, b]\) and is differentiable on \((a, b)\). If \( f(a) = f(b) \), then there exists a number \( c \) in \((a, b)\) such that \( f'(c) = 0 \).

Theorem **Mean Value Theorem**: Suppose that \( f \) is continuous in \([a, b]\) and is differentiable on \((a, b)\). Then there exists a number \( c \) in \((a, b)\) such that

\[
 f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Note that:

- **Rolle’s Theorem** is a special case of the **Mean Value Theorem**.

- By Rolle’s Theorem, we know if \( f'(x) \neq 0 \) for all \( x \) in \((a, b)\), then \( f(a) \neq f(b) \). On the other hand, the condition \( f(a) \neq f(b) \) alone is not enough for us to determine if there exists a number \( c \) in \((a, b)\) such that \( f'(c) = 0 \).

Graphically, Rolle’s Theorem and the Mean Value Theorem can be described as follows.

![Graphical illustration of Rolle’s Theorem and Mean Value Theorem](image)

- \( f(0) = f(2), \ f'(1) = 0 \)
- \( c \approx -1.4, \ c \approx 2.2 \)
- When \( b \) is close to \( a \) (\(|b - a|\) is small), \( f'(c) \approx \frac{f(b) - f(a)}{b - a} \) and \( f'(x) \approx \frac{f(b) - f(a)}{b - a} \) for \( x \in (a,b) \).

**Example** First show that the equation \( x - e^{-x} = 0 \) has a solution in \([0,1]\). Then Determine if the solution is unique.

Let \( f(x) = x - e^{-x} \). Since \( f(0)f(1) = (0 - 1)\left(1 - \frac{1}{e}\right) = -0.632 < 0 \), by the Intermediate Value Theorem there exists a number \( c \) in \((0, 1)\) such that \( f(c) = 0 \). Therefore, the equation \( x - e^{-x} = 0 \) has a solution in \([0, 1]\). Now let us check to see if \( f'(x) = 0 \) for \( x \) in \((0, 1)\),

\[
 f'(x) = 1 + e^{-x} > 0 \text{ for all } x.
\]

So, \( f'(x) \neq 0 \) for all \( x \) in \((0, 1)\) and therefore, \( f(x) = 0 \) only once and the solution is unique.
Example  The graph of \( f(x) = x \cos(x + 0.2) \) for \( x \) in \([-2, 2]\) is given below. Find graphically all possible \( c \) in \((-2, 2)\) satisfying the conclusion given in the Mean Value Theorem. Approximate \( f'(c) \) for each \( c \).

Approximately, \( c_1 = -1.19 \) and \( c_2 = 0.89 \).

\[
\begin{align*}
    f'(c_1) & \approx \frac{f(-1) - f(-1.5)}{-1 + 1.5} = \frac{-0.697 - (-0.401)}{0.5} = -0.592 \\
    f'(c_2) & \approx \frac{f(1) - f(0.5)}{1 - 0.5} = \frac{0.362 - 0.382}{0.5} = -0.04
\end{align*}
\]

Comparison: \( f'(c_1) = -0.446, \quad f'(c_2) = -0.327 \)

2. Fixed-Point of a Function:

Definition  A number \( p \) is said to be a fixed point of a function \( g(x) \) if \( g(p) = p \).

Graphically, a function has a fixed point at \( x = p \) if its graph \( y = g(x) \) intersects with the line \( y = x \) intersect at \( x = p \). For example, the following graphs show that \( y = e^{-x} \) and \( y = x \) intersect at a power where \( x \) is near 0.58.

\[
e^{-p} = p, \text{ when } p \approx 0.58
\]

Hence, \( f(x) = e^{-x} \) has a fixed point near 0.58. Some functions may have more than one fixed points and some functions may not have a fixed point. For example, the function in (i) has no fixed point because \( y = x^2 + 1 \) and \( y = x \) do not intersect; and the function in (ii) has infinitely many fixed points.
Algebraically, we solve the equation $g(x) = x$ (or $g(x) - x = 0$) to determine if a function has any fixed point over a given interval.

**Example** Find all fixed points of $g_1(x) = x^2 + 1$ and $g_2(x) = x + \cos(x)$ if they exist.

a. Set $g_1(x) = x$: $x^2 + 1 = x$, $x^2 - x + 1 = 0$. Using the quadratic formula:

$$x = \frac{1 \pm \sqrt{1-4(1)}}{2} = \frac{1 \pm \sqrt{-3}}{2}$$

no real solution.

So, $g_1(x)$ has no fixed point for $-\infty < x < \infty$.

b. Set $g_2(x) = x$: $x + \cos(x) = x$, $\cos(x) = 0$, $x = \pm \frac{2n-1}{2}\pi$, $n = 1, 2, 3, \ldots$

So, $g_2(x)$ has infinitely many fixed points for $-\infty < x < \infty$.

3. Existence and Uniqueness of a Fixed Point

**Theorem** Existence and Uniqueness: Let $g$ be continuous on $[a, b]$.

i. If $a \leq g(x) \leq b$ for all $x$ in $[a, b]$, then $g(x)$ has a fixed point $p$ in $[a, b]$.

ii. If, in addition, $g'(x)$ exists on $(a, b)$ and there exists a constant $0 < K < 1$ such that $|g'(x)| \leq K$ for all $x$ in $(a, b)$,

then $p$ is unique.

**Note that:**

a. Both conditions: $a \leq g(x) \leq b$ for all $x$ in $[a, b]$ and $|g'(x)| \leq K$ for all $x$ in $(a, b)$ are sufficient conditions. So, in the case where the condition in i. does not hold, it is possible that $g(x)$ has a fixed point; and in the case where the condition in ii. is not satisfied, it is also possible the fixed point of $g(x)$ is unique.

b. Because $g'(x)$ is the slope of the tangent line to the curve $y = g(x)$ at $x$, $|g'(x)| \leq K < 1$ means that the graph of $g(x)$ does not grow as faster than $y = x$ and not slower than $y = -x$.

**Proof of the existence and uniqueness:**

i. If $g(a) = a$ or $g(b) = b$, then $p = a$ or $p = b$ and $g$ has a fixed point. Now let $g(a) > a$ and $g(b) < b$, and let $h(x) = g(x) - x$. Since $g(x)$ is continuous on $[a, b]$, $h(x)$ is continuous on $[a, b]$. Observe that $h(a) = g(a) - a > 0$ and $h(b) = g(b) - b < 0$. So, by the Intermediate Value Theorem, we know there exists a number $c$ in $(a, b)$ such that $h(c) = 0$, that is
\( g(c) - c = 0 \) or \( g(c) = c \).

So, \( c \) is a fixed point of \( g \) in \([a, b]\).

ii. Now let also \( |g'(x)| \leq K \) for all \( x \) in \((a, b)\) where \( 0 < K < 1 \). Suppose that \( g(x) \) has two fixed points, say \( p < q \) in \([a, b]\). Then by the Mean Value Theorem, we know these exists a point \( c \) in \((q, p)\) such that

\[
g'(c) = \frac{g(p) - g(q)}{p - q}.
\]

Since \( g(p) = p \) and \( g(q) = q \), \( \frac{g(p) - g(q)}{p - q} = \frac{p - q}{p - q} = 1 \). So, \( g'(c) = 1 \), this contradicts the given condition \(|g'(x)| < 1 \) for all \( x \) in \((a, b)\). So, \( g \) cannot have two fixed points in \((a, b)\).

**Example** Let \( g(x) = \frac{1}{3}(x^2 - 1) \) for \( x \) in \([-1, 1]\). Determine if \( g \) has a fixed point in \([-1, 1]\). If so, determine if the fixed point is unique.

Check conditions given in the theorem for the existence and uniqueness:

i. Observe that \( g_{\text{min}} = g(0) = -\frac{1}{3} \geq -1 \) and \( g_{\text{max}} = g(1) = g(-1) = 0 \leq 1 \). Since \(-1 \leq g(x) \leq 1 \) for all \( x \) in \([-1, 1]\), \( g \) has a fixed point in \([-1, 1]\).

ii. Compute \( g'(x) = \frac{2}{3}x \). Since \( |g'(x)| = \frac{2}{3}|x| \leq \frac{2}{3} < 1 \), \( g \) has a unique fixed point in \([-1, 1]\).

For \( g(x) \), we can solve its fixed point \( p \) algebraically.

\[
x = \frac{1}{3}(x^2 - 1) = x^2 - 3x - 1 = 0 \Rightarrow x = \frac{3 \pm \sqrt{9 - 4(-1)}}{2} = \frac{3 \pm \sqrt{13}}{2}
\]

Since \( \frac{3 + \sqrt{13}}{2} > 1 \), \( p = \frac{3 - \sqrt{13}}{2} = -0.302776 \) is a unique fixed point in \([-1, 1]\).

Check the graph of \( g(x) \):

\[
- y = \frac{1}{3}(x^2 - 1), \quad \ldots \quad y = x, \quad y = -
\]

**Example** Let \( g(x) = 3^{-x} \) for \( x \) in \([0, 1]\). Determine if \( g \) has a fixed point in \([0, 1]\). If so, determine if the fixed point is unique.

Check conditions given in the theorem for the existence and uniqueness:

i. Observe that \( g_{\text{min}} = g(1) = 3^{-1} > 0 \), and \( g_{\text{max}} = g(0) = 1 \). Since \( 0 \leq g(x) \leq 1 \), \( g(x) \) has a fixed point in \([0, 1]\).

ii. Compute \( g'(x) = -3^{-x}\ln 3 \). Since \( |g'(0)| = \ln 3 > 1 \), there is no conclusion
about the uniqueness.
From the graph of $g$ below, we can see that $g$ has a unique fixed point $p \approx 0.55$ in $[-1, 1]$. But we cannot solve $p$ algebraically. How can solve a fixed point numerically?

$y = 3^{-x}, x$ in $[0, 1]$

4. The Fixed-Point Algorithm:
The Fixed-Point Algorithm is an algorithm that finds the fixed-point of a function over an interval assuming the fixed point is unique in this interval.

**Algorithm Fixed-Point Algorithm:** Given $g(x)$, and $[a, b]$, choose $p_0$ in $[a, b]$ and compute $p_1, p_2, \ldots$, as follows:

$$p_n = g(p_{n-1}) \quad \text{for} \quad n = 1, 2, \ldots$$

Implement the algorithm in a programming language which does the following:

1. Input $g(x)$, interval $[a, b]$, $p_0$ in $[a, b]$, $\epsilon$ and $K_{\text{max}}$, and compute $p_n = g(p_{n-1})$ for $n = 1, 2, \ldots$. The program terminates if
   i. $|p_n - p_{n-1}| < \epsilon$ and then $p \approx p_n$; or
   ii. $p_n > b$ or $p_n < a$, and the program fails; or
   iii. $n = K_{\text{max}}$.

The MatLab program fixpt.m implements the Fixed-Point Algorithm to find $p_n$ with input

1. the function gfun for $g(x)$;
2. an initial approximation $p_0$ to the fixed-point;
3. an accuracy requirement $\epsilon$; and
4. the maximum number for iterations $K_{\text{max}}$.

The following two examples show graphically how the Fixed-Point Algorithm works.
Fixed Point Iterations: \( g(x) = (x-0.5)^2, \ p_0 = 2 \)

\[ 0 \quad 1 \quad 1.5 \quad 2 \quad 2.5 \quad 3 \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

Clearly, the Fixed-Point Algorithm finds the fixed point \( p \) in (1) and diverges in (2).

5. Convergence and the Rate of Convergence of the Fixed-Point Algorithm:

Questions: Assume that \( g \) has a unique fixed point \( p \) in \( [a, b] \) and \( p_0 \) is in \( [a, b] \). Let

\[ p_n = g(p_{n-1}), \ n = 1, 2, \ldots. \]

a. Under what condition(s), does \( p_n \) converge to \( p \)?

b. If \( \lim_{n \to \infty} p_n = p \), what is the rate of converge?

Fixed-Point Theorem:

**Theorem** Fixed-Point Theorem: Let \( g \) be continuous on \( [a, b] \) and \( a \leq g(x) \leq b \). Suppose that \( g'(x) \) exists for all \( x \) in \( (a, b) \), and

\[ |g'(x)| \leq K \text{ for all } x \text{ in } (a, b) \text{ where } 0 < K < 1. \]

Then \( \lim_{n \to \infty} p_n = p \) for any \( p_0 \) in \( [a, b] \), and

\[ |p_n - p| \leq K^n \max \{ p_0 - a, b - p_0 \} \]

\[ |p_n - p| \leq \frac{K^n}{1 - K} |p_1 - p_0|, \text{ for all } n = 1, 2, \ldots. \]

**Proof:** Let \( p_0 \) be in \( [a, b] \) and \( \{p_n\} \) be generated by the Fixed-Point Algorithm. Observe that

\[ |p_n - p| = |g(p_{n-1}) - g(p)|. \]

By the Mean Value Theorem, we know there exists a number \( c \) in \( (a, b) \) such that

\[ \frac{g(p_{n-1}) - g(p)}{p_{n-1} - p} = g'(c) \text{ or } g(p_{n-1}) - g(p) = g'(c)(p_{n-1} - p). \]

Since \( |g'(x)| < 1 \) for all \( x \) in \( (a, b) \),
\[ |p_n - p| = |g(p_{n-1}) - g(p)| = |g'(c)(p_{n-1} - p)| = |g'(c)| |p_{n-1} - p| \leq K |p_{n-1} - p| \]
\[ \leq K(K |p_{n-2} - p|) = K^2 |p_{n-2} - p| \ldots \]
\[ \leq K^n |p_0 - p| \]
\[ 0 < \lim_{n \to \infty} |p_n - p| \leq \lim_{n \to \infty} K^n |p_0 - p| \Rightarrow 0. \]

Therefore, \( \lim_{n \to \infty} p_n = p. \)

Since \( |p_0 - p| \leq |p_0 - a| \) or \( |p_0 - p| \leq |b - p_0|, \)
\[ |p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \leq K |p_n - p_{n-1}| = K |g(p_{n-1}) - g(p_{n-2})| \]
\[ \leq K^2 |g(p_{n-2}) - g(p_{n-3})| \leq \ldots \]
\[ \leq K^n |p_1 - p_0| \]

So for any \( m > n \geq 1, \)
\[ |p_m - p_n| = |(p_m - p_{m-1}) + (p_{m-1} - p_{m-2}) + \ldots + (p_{n+1} - p_n)| \]
\[ \leq |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \ldots + |p_{n+1} - p_n| \]
\[ \leq K^{m-1} |p_1 - p_0| + K^{m-2} |p_1 - p_0| + \ldots + K^n |p_1 - p_0| \]
\[ = K^n (K^{m-n-1} + K^{m-n-2} + \ldots + K + 1) |p_1 - p_0| \]
\[ = K^n \left( \frac{1 - K^{m-n}}{1 - K} \right) |p_1 - p_0| \]

Since \( \lim_{m \to \infty} p_m = p, \)
\[ |p - p_n| = \lim_{m \to \infty} |p_m - p_n| \leq \lim_{m \to \infty} K^n \left( \frac{1 - K^{m-n}}{1 - K} \right) |p_1 - p_0| = \frac{K^n}{1 - K} |p_1 - p_0| \]

Note that:

a. **Rate of convergence:** \( |p_n - p| \leq \frac{K^n}{1 - K} |p_1 - p_0|, \) for all \( n = 1, 2, \ldots \) implies that
\[ \frac{|p_n - p|}{K^n} \leq \frac{1}{1 - K} |p_1 - p_0| = \frac{1}{1 - K} |g(p_0) - p_0|. \]

So \( p_n \) converges to \( p \) with the rate of convergence of \( O(K^n) \), i.e.,
\[ p_n = p + O(K^n). \]

b. **Order of convergence:** From \( |p_n - p| = |g'(c_{n-1})||p_{n-1} - p| \leq K |p_{n-1} - p|, \) we have
\[ \frac{|p_n - p|}{|p_{n-1} - p|} \leq K. \]

Hence, \( \{p_n\} \) converges to \( p \) linearly (\( \alpha = 1 \)) with an asymptote error constant \( K. \)

c. **The smallest possible number of iterations:** For a given \( \varepsilon, \) we can estimate the number \( N \) of iterations needed to approximate \( p \) by \( p_N. \) That is, find \( N \) such that
\[ \frac{K^N}{1 - K} |p_1 - p_0| < \frac{K^N}{1 - K} |b - a| < \varepsilon \Rightarrow K^N < \varepsilon \left( \frac{1 - K}{b - a} \right) \Rightarrow N \ln(K) < \ln\left( \varepsilon \left( \frac{1 - K}{b - a} \right) \right) \]

Since \( 0 < K < 1, \) \( \ln(K) < 0. \) So, \( N > \frac{\ln \varepsilon + \ln \left( \frac{1 - K}{b - a} \right)}{\ln(K)}. \)
Example  Determine whether or not the function has a fixed point in the given interval. If so, determine if the Fixed-point Algorithm will converge to the fixed point. In the case when it converges, estimate the number of iterations possibly needed to approximate the fixed point within $10^{-5}$.

\[ (1) \quad g(x) = \frac{1}{3} (2 - e^x + x^2), \quad [0, 1] \quad (2) \quad g(x) = \frac{1}{2} (10 - x^3)^{1/2}, \quad [0, 2] \]

(1) Check the range of $g$ : From the graph of $g(x)$ on $[0, 1]$, we have $0 \leq g(x) \leq 1$.

\[ (1) \quad y = g(x) = \frac{1}{3} (2 - e^x + x^2) \quad (2) \quad y = |g'(x)| = \left| \frac{1}{3} (-e^x + 2x) \right| \]

So, $g(x) = \frac{1}{3} (2 - e^x + x^2)$ has a fixed point in $[0, 1]$.

Check the maximum value of $|g'(x)|$ : $g'(x) = \frac{1}{3} (-e^x + 2x)$. From the graph of $|g'(x)|$,

\[ |g'(x)| \leq \left| g'(0) \right| = \left| \frac{1}{3} (-1) \right| = \frac{1}{3} = K < 1 \text{ for all } x \text{ in } [0, 1]. \]

So, $g(x)$ has a unique fixed-point in $[0, 1]$ and the sequence $\{p_n\}$ generated by the Fixed-Point Algorithm converges to $p$.

Estimate the number $N$ of iterations:

\[ N > \frac{\ln 10^{-5} + \ln \left( 1 - \frac{1}{3} \right)}{\ln \left( \frac{1}{3} \right)} = 10.8486, \quad \text{let } N = 11. \]

Use the Fixed-Point Algorithm to solve the fixed point in $[0, 1]$ and $p \approx p_n$ where $|p_n - p_{n-1}| < 10^{-5}$.

Using MatLab program fixpt.m, we have the following.

\[
\begin{align*}
&>> \text{gfun= @(x) } \frac{1}{3}*(2-\exp(x)+x.^2); \\
&>> \text{fixpt} \\
&\text{input initial point } p0 = 0 \\
&\text{input the tolerance for stopping criterion, ex:.0001,10^(-8) 10^(-5) } \\
&\text{input the maximum number of iterations 100} \\
&\text{Algorithm converges with number of iterations =} \\
&\text{ans = 9} \\
&\text{fixed point } p = \\
&0.257531806267540 \\
&\text{Now we check the values of } g(x) \text{ at } \{p_i\}: \\
&>> [p \ gfun(p)] \\
&0 \quad 0.333333333333333 \\
&0.333333333333333 \quad 0.238499562008341
\end{align*}
\]
As you see, \( g(p_9) \approx p_9 \).

(2) Check the range of \( g(x) \): Observe that

\[
g_{\text{min}} = g(2) = \frac{1}{2}\sqrt{10 - 8} = \frac{\sqrt{2}}{2} \geq 0, \quad g_{\text{max}} = g(0) = \frac{1}{2}\sqrt{10} \leq 2.
\]

So, \( 0 \leq g(x) \leq 2 \) for all \( x \) in \([0, 2]\). Hence, \( g(x) \) has a fixed point in \([0, 2]\).

Check the maximum value of \( |g'(x)| \):

\[
g'(x) = \frac{1}{2} \frac{-3x^2}{\sqrt{10 - x^3}}. \quad |g'(x)| = \frac{3x^2}{2\sqrt{10 - x^3}}
\]

From the graph of \( |g'(x)| \),

\[
y = |g'(x)| = \left| \frac{1}{2} \frac{-3x^2}{\sqrt{10 - x^3}} \right|
\]

\(|g'(x)| > 1 \) for some \( x \) in \([0, 2]\). So we cannot conclude the sequence \( \{p_n\} \) generated by the Fixed-Point Algorithm converges to \( p \).

**Using MatLab program fixpt.m, we have the following.**

```
>> gfun=@(x) 1/2*sqrt(10-x.^3);
>> fixpt
input initial point p0 = 0
input the tolerance for stopping criterion, ex:.0001, 10^(-8) 10^(-5)
input the maximum number of iterations 100
Algorithm converges with number of iterations = 18
fixed point p = 1.365227242758129
Check some values of \( p_i \) and \( g(p_i) \):
>> [p(13:18,1) gfun(p(13:18,1))]
1.365076127110441 1.365308786086649
1.365308786086649 1.365189681935091
1.365189681935091 1.365250660802594
```
6. Fixed-Point Algorithm for Solving The Equation: f(x) = 0

Let \( x^* \) be a solution of the equation \( f(x) = 0 \). To solve \( x^* \) using the Fixed-Point Algorithm, a function \( g \) needs to be defined first such that \( x^* \) is a fixed point of \( g \), that is, \( x^* = g(x^*) \).

**Example** Consider solving \( x^3 + x + 1 = 0 \). Find an interval \([a, b]\) on which the equation has a solution. Find a function \( g \) such that the fixed point of \( g \) is the solution of the equation: \( f(x) = 0 \). Determine if the sequence \( \{p_n\} \) generated by the Fixed-Point Algorithm with the function \( g \).

Consider \([a, b] = [-1, 0]\). Since \( f(-1)f(0) = (-1)(1) < 0 \), the equation has a solution in \([-1, 0]\).

(1) A naive choice of \( g \) : since \( x = -1 - x^3 \), we can let

\[
g(x) = -1 - x^3.
\]

Check the range of \( g \) : \( g_{\min} = g(0) = -1 \) and \( g_{\max} = g(-1) = 0 \), so, \(-1 \leq g(x) \leq 0 \) and \( g \) has a fixed point in \([-1, 0]\).

Check the maximum value of \( |g'(x)| \) : \( g'(x) = -3x^2 \), \( |g'(x)| = 3x^2 > 1 \) for some \( x \) in \([-1, 0]\).

So, it is not certain by the Fixed-Point Theorem if \( \{p_n\} \) converges to \( p \).

Observe that \( \{p_n\}_{n=0} = \{0, -1, 0, -1, \ldots\} \).

(2) Rewrite the equation \( x^3 + x + 1 = 0 \) as \( x^3 = -1 - x \), \( x = -\sqrt[3]{x + 1} \). Let

\[
g(x) = -\sqrt[3]{x + 1}.
\]

Check the range of \( g \) : \( g_{\min} = g(0) = -1 \), \( g_{\max} = g(-1) = 0 \), so, \(-1 \leq g(x) \leq 0 \) and \( g \) has a fixed point in \([-1, 0]\).

Check the maximum value of \( |g'(x)| \) : \( g'(x) = -\frac{1}{3} \frac{1}{\sqrt[3]{(x + 1)^2}} \). Since \( g'(x) \) is not defined at \( x = -1 \), \( g'(x) \) is unbounded. So, it is not certain if \( \{p_n\} \) converges to \( p \).

Observe that \( \{p_n\}_{n=0} = \{0, -1, 0, -1, \ldots\} \).

(3) Rewrite the equation \( x^3 + x + 1 = 0 \) as \( x^3 + x = -1 \) and then \( x(x^2 + 1) = -1 \) or \( x = \frac{-1}{x^2 + 1} \). Let

\[
g(x) = \frac{-1}{x^2 + 1}.
\]

Check the range of \( g \) : \( g_{\min} = g(0) = -1 \), and \( g_{\max} = g(-1) = -\frac{1}{2} \), so, \(-1 \leq g(x) \leq 0 \) and \( g \) has a fixed point in \([-1, 0]\).
Check the maximum value of $|g'(x)|$:

\[
\begin{align*}
g'(x) &= -\frac{2x}{(x^2 + 1)^2}, \\
|g'(x)| &= \frac{2|x|}{(x^2 + 1)^2} \quad \text{for } x < 0 \\
&= \frac{-2x}{(x^2 + 1)^2}.
\end{align*}
\]

From the graph of $|g'(x)|$ above, we see $|g'(x)| \leq 0.7 = K < 1$. So, $p$ is unique in $[-1, 0]$ and $\{p_n\}$ converges to $p$. Algebraically, let $h(x) = |g'(x)|$. $h'(x) = \frac{-2(1 - 3x^2)}{(x^2 + 1)^3} = 0$, $x = -\frac{1}{\sqrt{3}}$ for $x < 0$.

$h(x)$ reaches its maximum at $x = -\frac{1}{\sqrt{3}}$ and hence, $K = \frac{-2\left(-\frac{1}{\sqrt{3}}\right)}{\left(\left(-\frac{1}{\sqrt{3}}\right)^2 + 1\right)^2} = 0.64951905 < 0.65$

Estimate the number $N$ of iterations needed:

\[
N > \frac{\ln 10^{-5} + \ln(1 - 0.7)}{\ln(0.7)} = 35.654, \text{ let } N = 36.
\]

\[
N > \frac{\ln 10^{-5} + \ln(1 - 0.65)}{\ln(0.65)} = 29.162595, \text{ let } N = 30.
\]

Use the Fixed-Point Algorithm to solve the fixed point in $[0, 1]$ and $p \approx p_n$ where $|p_n - p_{n-1}| < 10^{-5}$.

$n = 27$ and $p_{27} = -0.68232442571947$.

Example Show that each of the following functions has a fixed point at $p$ precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

\[
g(x) = (3 + x - 2x^2)^{1/4} \quad \text{and} \quad g(x) = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}
\]

(1) Set $x^4 + 2x^2 - x - 3 = 0$. Then

\[
x^4 = 3 + x - 2x^2 \Rightarrow x = (3 + x - 2x^2)^{1/4}.
\]

(2) Check if $x = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$, then

\[
\begin{align*}
x - \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1} &= \frac{4x^4 + 4x^2 - x - (3x^4 + 2x^2 + 3)}{4x^3 + 4x - 1} \\
&= \frac{x^4 + 2x^2 - x - 3}{4x^3 + 4x - 1} = 0.
\end{align*}
\]

So, $x^4 + 2x^2 - x - 3 = 0$.

Example The following four methods are proposed to compute $7^{1/5}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.  

\[
\begin{array}{c|c|c|c|c}
\text{Method} & \text{Convergence} & \text{Rate of Convergence} & \text{Approximation} \\
\hline
\text{Newton-Raphson} & \text{Quadratic} & \text{Newton-Raphson} & 7^{1/5} \\
\text{Secant Method} & \text{Linear} & \text{Secant Method} & 7^{1/5} \\
\text{Bisection Method} & \text{Linear} & \text{Bisection Method} & 7^{1/5} \\
\text{Fixed-Point Method} & \text{Linear} & \text{Fixed-Point Method} & 7^{1/5}
\end{array}
\]
The function $g$ for each iteration is:

1. $g_1(x) = \left(1 + \frac{7-x^3}{x^2}\right)^{1/2} = \left(1 + \frac{7}{x^2} - x\right)^{1/2}$
2. $g_2(x) = x - \frac{x^5}{x^2} = x - x^3 + \frac{7}{x^2}$
3. $g_3(x) = x - \frac{x^5 - 7}{5x^4} = x - \frac{1}{5}x + \frac{7}{5x^4} = \frac{4}{5}x + \frac{7}{5x^4}$
4. $g_4(x) = x - \frac{x^5}{12} = x - \frac{x^5}{12} + \frac{7}{12}$

All four functions have a fixed point on $[1, 2]$. Compute $g'(x)$:

(i) $g_1'(x) = \frac{1}{2} \left(1 + \frac{7}{x^2} - x\right)^{-1/2} \left(-\frac{14}{x^3} - 1\right)$
(ii) $g_2'(x) = 1 - 3x^2 - \frac{14}{x^3}$
(iii) $g_3'(x) = \frac{4}{5} - \frac{28}{5x^5}$
(iv) $g_4'(x) = 1 - \frac{5}{12}x^4$

Check the value of $|g'(x)|$ at $p = 7^{1/5}$:

$|g_1'(p)| = \left|\frac{1}{2} \left(1 + \frac{7}{p^2} - p\right)^{-1/2} \left(-\frac{14}{p^3} - 1\right)\right| = 1.61828 > 1$

$|g_2'(p)| = \left|1 - 3p^2 - \frac{14}{p^3}\right| = 9.88953 > 1$

$|g_3'(p)| = \left|\frac{4}{5} - \frac{28}{5p^5}\right| = 0 < 1$

$|g_4'(p)| = \left|1 - \frac{5}{12}p^4\right| = 0.976365 < 1$
So, the sequences \( \{p_n\} \) generated by the Fixed-Point Algorithm using \( g_1 \) and \( g_2 \) do not converge. The sequences \( \{p_n\} \) generated by the Fixed-Point Algorithm using \( g_3 \) and \( g_4 \) converge and the third sequence converges faster than the the 4th one. The testing results show that

using \( g_3(x) \), \( 7^{1/5} \approx p_7 = 1.47577316159456 \)

using \( g_4(x) \), \( 7^{1/5} \approx p_{355} = 1.47577807080213 \)

**Example**  
Show that if \( A \) is any positive number, then the sequence defined by

\[
x_n = \frac{1}{2} x_{n-1} + \frac{A}{2x_{n-1}}, \text{ for } n \geq 1,
\]

converges to \( \sqrt{A} \) whenever \( x_0 > 0 \).

Let \( \lim_{n \to \infty} x_n = x \). Then

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} \left( \frac{1}{2} x_{n-1} + \frac{A}{2x_{n-1}} \right) \iff x = \frac{1}{2} x + \frac{A}{2x} \iff \frac{1}{2} x = \frac{A}{2x} \iff x^2 = A \iff x = \sqrt{A}.
\]

**Example**  
Let \( \{p_n\} \) be generated by the **Fixed-point Algorithm** with the function \( g(x) \) and let \( p \) be the fixed point of \( g(x) \) such that \( \lim_{n \to \infty} p_n = p \). Determine the orders of convergence and the asymptotic error constants of the sequence \( \{p_n\} \) in the cases where

(1) \( g'(p) \neq 0 \); and

(2) \( g'(p) = 0 \) but \( g''(p) \neq 0 \).

(1) Assume that \( g'(p) \neq 0 \). By the Mean Value Theorem, we have

\[
| p_{n+1} - p | = | g(p_n) - g(p) | = | g'(c_n)(p_n - p) | = | g'(c_n) || p_n - p |
\]

where \( c_n \) is between \( p_n \) and \( p \).

\[
\lim_{n \to \infty} | p_{n+1} - p | = \lim_{n \to \infty} | g'(c_n) | = | g'(p) |.
\]

So, the order of convergence is 1 and the asymptotic error constant \( |g'(p)| \).

(2) Assume that \( g'(p) = 0 \) but \( g''(p) \neq 0 \). By the Taylor Theorem, we have for \( x \neq p \),

\[
g(x) = g(p) + \frac{g'(p)}{1!} (x - p) + \frac{g''(\xi(x))}{2!} (x - p)^2, \text{ where } \xi(x) \text{ is between } x \text{ and } p.
\]

Then

\[
g(p_n) = g(p) + \frac{g'(p)}{1!} (p_n - p) + \frac{g''(\xi(x_n))}{2!} (p_n - p)^2 = g(p) + \frac{g''(\xi(p_n))}{2!} (p_n - p)^2
\]

\[
| p_{n+1} - p | = | g(p_n) - g(p) | = | g(p) + \frac{g''(\xi(p_n))}{2!} (p_n - p)^2 - g(p) | = \frac{1}{2} | g''(\xi(p_n)) || p_n - p |^2
\]

\[
\frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{1}{2} | g''(\xi(p_n)) |, \quad \lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{1}{2} | g''(p) |
\]

So, the order of convergence is 2 and the asymptotic error constant \( \frac{1}{2} |g''(p)| \).
Exercises:

1. The graph of $f(x)$ for $-3 \leq x \leq 5$ is given at the left.
    Find graphically all possible $c$ in $(-3, 5)$ that satisfies the conclusion given by the Mean Value Theorem.

2. The graph of $g$ for $x$ in $[0, 1]$ is given at the left. Let $p_0 = 0$. Compute graphically $p_1$, $p_2$, $p_3$ generated by the Fixed-point Algorithm.

3. Use the Fixed-Point Algorithm to approximate the solution of the equation $x^3 - x - 1 = 0$ on $[1, 2]$ within $10^{-5}$ with $p_0 = 1$. Give $g(x)$ first and explain how you choose $g(x)$.

4. Consider solving the equation $\cos(x) - x = 0$ for $x$ in $[0, \frac{\pi}{2}]$.
   a. Show that the equation $\cos(x) - x = 0$ has a unique solution in $[0, \frac{\pi}{2}]$ by two steps:
      (i) show the equation has a solution in $[0, \frac{\pi}{2}]$ by the Intermediate Value Theorem;
      (ii) show the solution is unique by Rolle’s Theorem.
   b. Approximate the solution of the equation within $10^{-8}$ by the following methods:
      (i) the Bisection Method ($\text{bisect.m}$);
      (ii) the Newton Method ($\text{newton.m}$) with $p_0 = 0$;
      (iii) the Fixed Point Method ($\text{fixpt.m}$) with $g(x) = \cos(x)$ and $p_0 = 0$.
      Report the number of iterations for each method. Rank the methods based on numbers of iterations.
   c. Estimate (the best you can) the asymptotic error constant $\lambda$ for the Newton Method. Does $\lambda$ match with the performance of the Newton Method?
5. Consider the function \( g(x) = 1 + x - \frac{1}{8}x^3 \).
   a. Show that \( g(x) \) has a unique fixed point on the real line.
   b. Can we show \( g(x) \) has a unique fixed point using the theorem for the existence and uniqueness? Explain.
   c. What is the order of convergence if we use the Fixed-Point Algorithm to find this fixed-point? Show your work in detail.

6. Consider the function \( g(x) = e^{-x^2} \).
   a. Show that \( g \) has a unique fixed point on the interval \([0, 1]\).
   b. Approximate the fixed point of \( g(x) \) within \( 10^{-8} \) using the Fixed-Point Algorithm.
   c. Estimate (algebraically) the number of iterations to approximate the fixed point within \( 10^{-8} \) (using the formula given in Notes c. after the Fixed-Point Theorem).

7. The following four methods are proposed to compute \( \sqrt[3]{21} \). Rank them in order, based on their apparent speed of convergence, assuming \( p_0 = 1 \).

   (1) \( p_n = \frac{20p_{n-1} + \frac{21}{p_{n-1}^2}}{21} \)

   (2) \( p_n = p_{n-1} - \frac{p_{n-1}^3 - 21}{3p_{n-1}^2} \)

   (3) \( p_n = p_{n-1} - \frac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21} \)

   (4) \( p_n = \sqrt[3]{\frac{21}{p_{n-1}}} \)