The Superiority of a New Type (2,2)-Step Iterative Method over the Related Chebyshev Method

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Abstract

A new type (2,2)-step iterative method related to an optimal Chebyshev method is developed for solving real and nonsymmetric linear systems of the form \( Ax = b \). It is an extension of the (2,2)-step iterative method introduced in [7, 9]. The superiority of the new type (2,2)-step iterative method over the optimal Chebyshev method is derived in the case where the known (2,2)-step iterative method may not improve the asymptotic rate of convergence. Two numerical examples are given to illustrate the results.

1 Introduction

Let \( Ax = b \) be a system of linear equations where \( A \) is real and nonsingular. The system can be written as an equivalent one in a \( \times \)xed point form
\[
x = Tx + c
\]
by a regular splitting. Suppose that the spectrum \( \sigma(T) \) of \( T \) is a subset of a closed ellipse \( \Omega \) excluding 1. \( \Omega \) is called an optimal ellipse in the sense that the Chebyshev iterative method [6, 14] determined by the foci of the boundary of \( \Omega \) for solving (1.1) is asymptotically optimal.

Manteuffel [10] has developed an adaptive procedure for estimating the foci of the optimal ellipse whose major axis either lies on the real axis or is parallel to the imaginary axis based on the power method. This adaptive dynamic scheme was modi\textit{ñ}ed based on the GMRES Algorithm by Elman et al [3] and further developed by Golub et al [1, 5] with an application of the modi\textit{ñ}ed moments. For simplicity we only consider the case where the major axis lies on the real axis since the results for the other case can be obtained in a similar way.

Let \( \partial \Omega \) be the boundary of \( \Omega \) and let the exterior of the ellipse \( \partial \Omega \) be the image of a scaled and translated Joukowsky transformation
\[
z = \Psi(w) := a \left( \frac{b}{w} \right) + d, \quad |w| > 1,
\]

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where

\[ a > 0, \quad 0 < b < 1 \quad \text{and} \quad a(1 + b) + d < 1. \tag{1.3} \]

Then \( \Psi \) maps the exterior of the unit circle in the extended \( w \)-plane 1-1 onto the exterior of \( \partial \Omega \) in the extended \( z \)-plane with \( \Psi(\infty) = \infty \) and \( \Psi'(\infty) > 0 \). It is known that the stationary 2-step iterative method generated by \( \Psi \) in (1.2) achieves the same asymptotically optimal rate of convergence of the Chebyshev method. The asymptotic rate of convergence of the optimal 2-step iterative method is the same as the asymptotic convergence factor (ACF) for \( \Omega \), denoted by \( \kappa(\Omega) \) and given by [2, 12]

\[ \kappa(\Omega) = \frac{1}{|\Psi^{-1}(1)|}. \]

Li [7] has investigated a (2,2)-step iterative method generated by the functions \( \Psi_c \) which is extended from \( \Psi \) with introducing a real parameter \( c \),

\[ \Psi_c(w) := a \left(w + \frac{b}{w-c}\right) + d - \frac{abc}{1-c}, \quad |w| > 1, \tag{1.4} \]

where \( a, b \) and \( d \) are the same as in (1.3). The relation of \( \Psi_c \) and \( \Psi \) and properties of \( \Psi_c \) are studied in [7, 8, 9] and summarized as follows.

**Theorem 1.1** [7] For each \( c \) with \( 0 < |c| < 1 \), the function \( \Psi_c \) is 1-1 on the exterior of the unit circle if and only if \(- (1 - c^2) \leq b \leq (1 - |c|)^2\). Moreover, let \( \Omega_c \) be the compact set bounded by the close curve \( \{z = \Psi_c(w) : |w| = 1\} \). Then the ACF for \( \Omega_c, \kappa(\Omega_c) \), is a monotonically decreasing function in \( c \) on \([- (1 - \sqrt{b}), 1 - \sqrt{b}] \).

The (2,2)-step iterative method corresponding to \( \Psi_c \) applied to (1.1) possesses the asymptotic rate of convergence

\[ \kappa(\Omega_c) = \frac{1}{|\Psi_c^{-1}(1)|}, \]

provided that \( \sigma(T) \subset \Omega_c \). Since we are only interested in the case where \( \kappa(\Omega_c) < \kappa(\Omega) \), it is natural to assume that

\[ 0 < c \leq 1 - \sqrt{b}. \tag{1.5} \]

**Theorem 1.2** [8] There exists a unique intersection point \( z_c \) of \( \partial \Omega \) and \( \partial \Omega_c \) on the upper half-plane for \( c \in (0, 1 - \sqrt{b}] \).

**Theorem 1.3** [9] If \( b < \sqrt{5} - 2 \), then there exists a unique \( c_1 \in (0, 1 - \sqrt{b}] \) such that \( z_{c_1} = d + ia(1 - b) \) is an intersection point of \( \partial \Omega \) and \( \partial \Omega_{c_1} \). Furthermore, if there is no eigenvalue of \( T \) in the 1st quadrant of \( \partial \Omega \) then there exists a \( c \in (0, c_1] \) such that \( \sigma(T) \subset \Omega_c \) and \( \kappa(\Omega_c) < \kappa(\Omega) \), and therefore, the (2,2)-step iterative method generated by \( \Psi_c \) improves the asymptotic rate of convergence of the optimal Chebyshev method.
Note that $b < \sqrt{5} - 2$ is equivalent to $\frac{1 - b}{1 + b} > \frac{\sqrt{5} - 1}{2}$, i.e., the ratio of the minor axis to the major axis of an optimal ellipse is greater than the golden ratio.

**Theorem 1.4** [9] If $b \geq \sqrt{5} - 2$, then $\text{Re}(z_c) < d$ for all $c \in (0, 1 - \sqrt{b})$ where $z_c$ is the unique intersection point of $\partial \Omega$ and $\partial \Omega_c$ on the upper half-plane.

In the case where $b \geq \sqrt{5} - 2$, as given in Theorem 1.4, the intersection point $z_c$ of $\partial \Omega$ and $\partial \Omega_c$ on the upper half-plane for all $c \in (0, 1 - \sqrt{b})$ must be in the second quadrant. Since $d + ia(1 - b) \notin \Omega_c$, the improvement of the asymptotic rate of convergence of the Chebyshev iterative method optimal for $\Omega$ for solving (1.1) remains unknown. Is there another type (2,2)-step iterative method which is better than the Chebyshev iterative method optimal for $\Omega$? This question is addressed throughout this paper. In Section 2, notations and assumptions are first introduced and a new type (2,2)-step iterative method is then presented.

This new method is generated by the function $\Psi_{c,f}$ which is extended from $\Psi_c$ with an additional parameter $f$,

$$\Psi_{c,f}(w) := a(w + \frac{f}{w - c}) + d + ab - \frac{af}{1 - c}, \quad |w| > 1. \tag{1.6}$$

Let $\partial \Omega_{c,f}$ be the close curve defined by $\{z = \Psi_{c,f}(w) : |w| = 1\}$. The function $\Psi_{c,f}$ has the same properties at infinity as $\Psi$ and $\Psi_c$ have: $\Psi_{c,f}(\infty) = \infty$ and $\Psi'_{c,f}(\infty) > 0$. The relation of $\partial \Omega$ and $\partial \Omega_{c,f}$ and the existence and uniqueness of the intersection point of $\partial \Omega$ and $\partial \Omega_{c,f}$ on the upper half-plane are studied in Section 3. In Section 4, a specific $\Psi_{c,f}$ is constructed for a $\partial \Omega$ determined by (1.2) when $b \geq \sqrt{5} - 2$ such that the vertex $d + ia(1 - b)$ of $\partial \Omega$ on the upper half-plane is an intersection point of $\partial \Omega$ and $\partial \Omega_{c,f}$. The superiority of the (2,2)-step method generated by $\Psi_{c,f}$ is given in Section 5. Examples are given in Section 6 to illustrate the results.

### 2 Preliminary

Consider the (2,2)-step iterative method generated by $\Psi_{c,f}$ in (1.6). The last term $(ab - \frac{af}{1 - c})$ in (1.6) is added to ensure that $\Psi_{c,f}(1) = \Psi(1)$, which allows us to compare the asymptotic rates of convergence of the iterative methods generated by $\Psi$ and $\Psi_{c,f}$, respectively. The following notations and assumptions are used throughout the paper.

(a) Notations:

(i) The boundaries of $\Omega$ and $\Omega_{c,f}$ are represented, respectively, as follows.

$$\partial \Omega = \{x + iy := \Psi(w) \mid w = e^{it} \text{ for } 0 \leq t < 2\pi\}$$

$$\partial \Omega_{c,f} = \{\mu + iv := \Psi_{c,f}(w) \mid w = e^{i\zeta} \text{ for } 0 \leq \zeta < 2\pi\} \tag{2.1}$$
(ii) For simplicity, we consider $\partial\Omega$ and $\partial\Omega_{cf}$ in (2.1) on the upper half-plane. Then the real and the imaginary parts of $\partial\Omega$ and $\partial\Omega_{cf}$ are given, respectively, by

$$
\begin{align*}
x(t) &= a(1+b)\cos t + d \\
y(t) &= a(1-b)\sin t
\end{align*}
$$

(2.2)

for $t \in [0,\pi]$, and

$$
\begin{align*}
\mu(s) &= a\left( s + \frac{f(s-c)}{1-2cs+c^2} \right) + d + ab - \frac{af}{1-c} \\
\nu(s) &= a\sqrt{1-s^2}\left( 1 - \frac{f}{1-2cs+c^2} \right)
\end{align*}
$$

(2.3)

for $\zeta \in [0,\pi]$ where $s = \cos \zeta$.

(iii) The (2,2)-step iterative method corresponding to $\Psi_{cf}$ applied to (1.1) possesses the asymptotic rate of convergence

$$
\kappa(\Omega_{cf}) = \frac{1}{|\Psi_{cf}^{-1}(1)|},
$$

(2.4)

provided that $\sigma(T) \subset \Omega_{cf}$.

(iv) The following notations are used:

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_e$</td>
<td>the intersection point of $\partial\Omega$ and $\partial\Omega_e$ on the upper half-plane</td>
</tr>
<tr>
<td>$z_{cf}$</td>
<td>the intersection point of $\partial\Omega$ and $\partial\Omega_{cf}$ on the upper half-plane</td>
</tr>
<tr>
<td>$f^*$</td>
<td>the value of $f$ such that $z_{cf} = d + ia(1-b)$ for a given $c$</td>
</tr>
<tr>
<td>$f_1$</td>
<td>the value of $f$ such that $\kappa(\Omega_{cf}) = \kappa(\Omega)$ for a given $c$</td>
</tr>
<tr>
<td>$\rho(c,f)$</td>
<td>the unique solution to $\Psi_{cf}(w) = 1$ for $w$</td>
</tr>
</tbody>
</table>

(b) Assumptions:

(i) We assume that the elliptic domain $\Omega$ contains $\sigma(T)$ and the three vertices of $\Omega$ are eigenvalues of $T$. Moreover, we assume that there is no eigenvalue of $T$ on $\partial\Omega$ in the first quadrant. In other words,

$$
\{\Psi(1), \Psi(\pm i)\} \subset \sigma(T) \subset \Omega \quad \text{and} \quad \{\Psi(e^{it}) \mid 0 < t < \pi/2\} \cap \sigma(T) = \emptyset.
$$

(2.5)

(ii) In addition to the conditions given in (1.3) and (1.5), we require

$$
b(1-c^2) \leq f \leq (1-c)^2.
$$

(2.6)

As the same one that for $\Psi_e$ given in Theorem 1.1, $\Psi_{cf}$ is a 1-1 mapping from the exterior of the unit circle onto the exterior of $\partial\Omega_{cf}$ if and only if $-(1-c^2) \leq f \leq (1-|c|)^2$. Since we consider only the case where $0 < c < 1$ in this paper, it is natural to assume $f \leq (1-c)^2$. On the other
hand, in order to have \( \Psi(-1) \) contained in \( \Omega_{cf} \) in the case where \( \Psi(-1) \) is an eigenvalue of \( T \), the following inequality must hold:

\[
d - a(1 + b) \geq \Psi_{cf}(-1) = -a(1 - b) - \frac{2af}{1 - c^2} + d,
\]

or

\[
\frac{2f}{1 - c^2} \geq -(1 - b) + (1 + b) = 2b,
\]

which is equivalent to \( b(1 - c^2) \leq f \).

3 The Intersection Point of \( \partial \Omega \) and \( \partial \Omega_{cf} \)

In this section, the relation of \( \partial \Omega \) and \( \partial \Omega_{cf} \) and the intersection point \( z_{cf} \) of \( \partial \Omega \) and \( \partial \Omega_{cf} \) with \( \text{Im}(z_{cf}) > 0 \) are studied. Without loss of generality, assume in this section that \( a = 1 \) and \( d = 0 \).

Let \( \text{Ext}(\partial \Omega) \) and \( \text{Int}(\partial \Omega) \) be the exterior and interior of the ellipse \( \partial \Omega \), respectively.

**Theorem 3.1** For \( f \in \left[ b(1 - c^2), \min\{b(1 + c^2), (1 - c)^2\} \right] \), there exists a unique intersection point

\[
z_{cf} = \Psi(e^{it}) = \Psi_{cf}(e^{i\varsigma})
\]

of \( \partial \Omega \) and \( \partial \Omega_{cf} \) with \( \text{Im}(z_{cf}) > 0 \) where \( s^* = \cos \varsigma \in (0, c) \).

**Proof.** Let \( z_{cf} = \Psi(e^{it}) = \Psi_{cf}(e^{i\varsigma}) \), for some \( t, \varsigma \in (0, \pi) \). From (2.2) and (2.3),

\[
\begin{cases}
\mu(s) = x(t) \\
\nu(s) = y(t)
\end{cases}
\]

implies

\[
\begin{cases}
s + \frac{f(s - c)}{1 - 2cs + c^2} + b - \frac{f}{1 - c} = (1 + b) \cos t \\
\sqrt{1 - s^2} \left(1 - \frac{f}{1 - 2cs + c^2}\right) = (1 - b) \sin t
\end{cases},
\]

where \( s = \cos \varsigma \). The following equation is derived by applying the identity \( \sin^2 t + \cos^2 t = 1 \) to above system:

\[
p(s) := \left(s + \frac{f(s - c)}{1 - 2cs + c^2} + b - \frac{f}{1 - c}\right)^2 \frac{1}{(1 + b)^2} + (1 - s^2) \left(1 - \frac{f}{1 - 2cs + c^2}\right)^2 \frac{1}{(1 - b)^2} - 1 = 0.
\]

Observe that \( p(s) \) has the following properties:

(i) \( p(s) > 0 \) if and only if \( \mu(s) + i\nu(s) \in \text{Ext}(\partial \Omega) \);

(ii) \( p(s) < 0 \) if and only if \( \mu(s) + i\nu(s) \in \text{Int}(\partial \Omega) \); and

(iii) \( p(s) = 0 \) if and only if \( \mu(s) + i\nu(s) \) on \( \partial \Omega \).
Based on these properties, it suffices to show that

Case 1. \( p(s) > 0 \) if \( s \in (-1, 0] \),
Case 2. \( p(s) \) has exactly one zero in \((0, c)\), and
Case 3. \( p(s) < 0 \) if \( s \in [c, 1) \).

Case 1. Let \( s \in (-1, 0] \). Note that for \( \mu(s) = x(t) \), \( p(s) > 0 \) if and only if \( \nu(s) > y(t) \). Assume that
\[
\mu(s) = x(t) \text{ for some } t \in (0, \pi].
\]
Since \( f \leq b(1 + c^2) \), it is true that
\[
\nu(s) = \sqrt{1 - s^2} \left(1 - \frac{f}{1 - 2cs + c^2}\right) > \sin t \left(1 - \frac{f}{1 + c^2}\right) \geq (1 - b) \sin t = y(t)
\]
if \( \sqrt{1 - s^2} \sin t \). Now we prove
\[
\sqrt{1 - s^2} > \sin t
\]
by showing that \(|\cos t| > |s| \). Since \( f \geq b(1 - c^2) \),
\[
\frac{f(1 + c)(1 - s)}{(1 - c)(1 - 2cs + c^2)} - b > \frac{f(1 + c)(1 - s)}{(1 - c)(1 - 2cs + c^2)} - \frac{f}{1 - c^2}
\]
\[
= \frac{(2c - sc^2 - s) f}{(1 - c^2)(1 - 2cs + c^2)} > 0.
\]
Then from the 1st equation of (3.1),
\[
(1 + b)|\cos t| = \left| s + \frac{f(s - c)}{1 - 2cs + c^2} + b \frac{f}{1 - c} \right| = \left| s - \frac{f(1 + c)(1 - s)}{(1 - c)(1 - 2cs + c^2)} + b \right|
\]
\[
= -s + \frac{f(1 + c)(1 - s)}{(1 - c)(1 - 2cs + c^2)} - b = (-s) + \frac{f(1 + c)(1 - s)}{(1 - c)(1 - 2cs + c^2)} - b
\]
\[
\geq (-s) + \frac{b(1 + c)^2(1 - s)}{1 + 2c(-s) + c^2} - b > (-s) + b(1 - s) - b = (-s)(1 + b)
\]
So,
\[
|\cos t| > -s = |s| \text{ for } s \in (-1, 0].
\]

Case 2. Let \( s \in (0, c) \). Define
\[
q(s) := (1 - 2cs + c^2)^2 p(s).
\]
Note that \( q(s) \) is a polynomial of degree 4 in \( s \). So the equation \( q(s) = 0 \) has at most 4 real solutions.
Note also that \( q(s) \) and \( p(s) \) have the same sign for all \( s \) and that \( q(s) = 0 \) if and only if \( p(s) = 0 \) for \( s \in [0, 1] \). So, we show that \( q(s) = 0 \) has a unique solution in \((0, c)\). Since
\[
\lim_{s \to -\infty} p(s) = \lim_{s \to -\infty} s^2 \left( \frac{1}{(1 + b)^2} - \frac{1}{(1 - b)^2} \right) = -\infty,
\]
\[
\lim_{s \to \pm\infty} q(s) = -\infty.
\]

Now we show that (a) \( \mu(0) + iv(0) \in \operatorname{Ext}(\partial \Omega) \), (b) \( \mu(c) + iv(c) \in \operatorname{Int}(\partial \Omega) \) and (3) \( \mu(1) + iv(1) \) on \( \partial \Omega \).

(a) Clearly, \( \mu(s) \) and \( \nu(s) \) in (2.3) are continuous. When \( f < b(1 + c^2) \),

\[
\nu(0) = 1 - \frac{f}{1 + c^2} > 1 - b.
\]

When \( f = b(1 + c^2) \),

\[
\mu(0) = -\frac{fc}{1 + c^2} + b - \frac{f}{1 - c} = b - f \frac{1 + c}{(1 + c^2)(1 - c)} = b - b \left( \frac{1 + c}{1 - c} \right) < 0
\]

and

\[
\nu(0) = 1 - b.
\]

Hence, \( \mu(0) + iv(0) \in \operatorname{Ext}(\partial \Omega) \).

(b) Since \( b(1 - c^2) \leq f \leq \min\{b(1 + c^2), (1 - c)^2\} \) and \( 0 \leq c \leq 1 - \sqrt{b} \), we have the following inequalities

\[
b(1 + c) \leq \frac{f}{1 - c} \leq 1 - c, \quad b(1 - c^2) \leq f \leq b(1 + c^2), \quad \text{and} \quad 0 < b \leq (1 - c)^2.
\]

Then

\[
\mu(c) = c + b - \frac{f}{1 - c} \leq c + b - b(1 + c) = c - bc < c,
\]

and

\[
\mu(c) \geq c + b - b \frac{1 + c^2}{1 - c} = \frac{c + b - c^2 - bc - b - bc^2}{1 - c} = \frac{c(1 - c - b(1 + c))}{1 - c} \geq \frac{c(1 - c - (1 - c)^2)(1 + c)}{1 - c} = c(1 - (1 - c^2)) > 0.
\]

Since \( 0 < \mu(c) < c < 1 + b \), there exists a unique \( t_1 \in (0, \pi/2) \) such that

\[
\mu(c) = x(t_1) = (1 + b) \cos t_1.
\]

Then (3.3) implies

\[
0 < \cos t_1 = \frac{\mu(c)}{1 + b} < \frac{c}{1 + b} < c,
\]

and

\[
\gamma(t_1) = (1 - b) \sin t_1 > (1 - b) \sqrt{1 - c^2} \geq \left( 1 - \frac{f}{1 - c^2} \right) \sqrt{1 - c^2} = \nu(c).
\]

Hence, \( \mu(c) + iv(c) \in \operatorname{Int}(\partial \Omega) \).

(c) \( \mu(1) + iv(1) = 1 + \frac{f(1 - c)}{1 - 2c + c^2} + b - \frac{f}{1 - c} = 1 + b \) on \( \partial \Omega \).
Since $\mu(0) + i\nu(0) \in \text{Ext}(\partial\Omega)$, $\mu(c) + iv(c) \in \text{Int}(\partial\Omega)$ and $\mu(1) + iv(1)$ on $\partial\Omega$, we have

$$p(0) > 0, \quad p(c) < 0 \quad \text{and} \quad p(1) = 0,$$

respectively, by the properties of $p(s)$. Hence,

$$q(0) > 0, \quad q(c) < 0, \quad \text{and} \quad q(1) = 0. \quad (3.4)$$

Results in $(3.2)$ and $(3.4)$ imply $q(s) = 0$ has at most two real solutions in $(0, c)$. However, the fact that $q(s)$ has apposite signs at $s = 0$ and $s = c$ implies $q(s) = 0$ has only one real solution in $(0, c)$. Therefore, $p(s) = 0$ has a unique solution on $(0, c)$.

Case 3. Let $s \in [c, 1]$. Note that for $\mu(s) = x(t)$, $p(s) < 0$ if and only if $\nu(s) < y(t)$. Assume that

$$\mu(s) = x(t) \quad \text{for some} \quad t \in (0, \pi].$$

Observe that

$$\nu(s) = \sqrt{1 - s^2} \left(1 - \frac{f}{1 - 2cs + c^2}\right) < \sqrt{1 - \cos^2 t} \left(1 - \frac{f}{1 - 2cs + c^2}\right)$$

$$< (1 - b) \sin t = y(t)$$

provided that $0 < \cos t < s$ for $s \in [c, 1]$. Now we show $0 < \cos t < s$ for $s \in [c, 1]$.

$$\cos t < s \iff (1 + b) \cos t < (1 + b) s$$

$$\iff s - \frac{f(1 + c)(1 - s)}{(1 - c)(1 - 2cs + c^2)} + b < (1 + b) s$$

$$\iff - \frac{f(1 + c)(1 - s)}{(1 - c)(1 - 2cs + c^2)} + b < bs.$$ 

Since $b\left(1 - c^2\right) \leq f$,

$$- \frac{f(1 + c)(1 - s)}{(1 - c)(1 - 2cs + c^2)} + b \leq b - \frac{b(1 + c)^2(1 - s)}{1 - 2cs + c^2} \leq b - \frac{b(1 + c)^2(1 - s)}{1 - c^2}$$

$$= b - \frac{b(1 + c)(1 - s)}{(1 - c)} < b - b(1 - s) = bs.$$ 

Hence, 

$$\cos t < s. \quad (3.5)$$

Since

$$\frac{d\mu}{ds} = 1 + \frac{f(1 - c^2)}{(1 - 2cs + c^2)^2} > 0,$$

$\mu(s)$ is strictly increasing and therefore,

$$x(t) = \mu(s) \geq \mu(c) = x(t_1)$$

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Lemma 4.1 studied.

For any

\[ \cos t \geq \cos t_1 > 0. \]  

(3.6)

Inequalities (3.5) and (3.6) imply

\[ 0 < \cos t < s. \]

Note that the condition

\[ f \in [b(1 - c^2), \min\{b(1 + c^2), (1 - c)^2\}] \]  

(3.7)

is stronger than (2.6). The relation of the ellipse \( \partial \Omega \) and the curve \( \partial \Omega_{cf} \) for \( c \in (0,1 - \sqrt{b}] \) and \( f \in [b(1 - c^2), \min\{b(1 + c^2), (1 - c)^2\}] \) is summarized in the following theorem.

**Theorem 3.2** (a) For \( f \in [b(1 - c^2), \min\{b(1 + c^2), (1 - c)^2\}] \), \( \partial \Omega \) and \( \partial \Omega_{cf} \) intersect at exact three points: \( \Psi(1) = \Psi_{cf}(1) \), and a pair of conjugate complex number \( z_{cf} = \Psi(e^{it}) = \Psi_{cf}(e^{i\xi^*}) \) and \( \Omega_{cf} \), with \( \text{Im}(z_{cf}) > 0 \) and \( \zeta^* \in (\cos^{-1}(c), \pi/2) \).

(b) The boundary of \( \Omega \cap \Omega_{cf} \) is given by

\[ \{z = \Psi(e^{it}) | t^* \leq t \leq 2\pi - t^*\} \cup \{z = \Psi_{cf}(e^{i\xi}) | -\zeta^* \leq \xi \leq \zeta^*\}. \]

Note that the location of \( z_{cf} \) is critical for determining whether a (2,2)-step iterative method generated by \( \Psi_{cf} \) converges faster than the optimal Chebyshev method.

4 A \( \Psi_{cf*} \) for a Given Optimal Ellipse \( \Omega \)

With the assumption for \( \sigma(T) \) in (2.5) and \( b \geq \sqrt{\frac{3}{2}} - 2 \), is there always a (2,2)-step iterative method generated by \( \Psi_{cf} \) which converges faster than the optimal Chebyshev method? Based on the results in Section 3 and Theorem 1.3, we need to And out And if there is always an \( f \in [b(1 - c^2), \min\{b(1 + c^2), (1 - c)^2\}] \) such that the vertex \( d + ia(1-b) \) of \( \partial \Omega \) is the intersection point \( z_{cf} \) of \( \partial \Omega \) and \( \partial \Omega_{cf} \) with \( \text{Im}(z_{cf}) > 0 \). In this section, we show that such an \( f \) exists, say \( f^* \). A formula for \( f^* \) is also derived and its properties are studied.

**Lemma 4.1** For any \( c \in (0,1 - \sqrt{b}] \), there exists an \( f \) such that the point \( z_{cf} = d + ia(1-b) \) is the intersection point of \( \partial \Omega \) and \( \partial \Omega_{cf} \) with \( \text{Im}(z_{cf}) > 0 \).

**Proof.** From (3.1), it is equivalent that \( z_{cf} = d + ia(1-b) \) is the intersection point of \( \partial \Omega \) and \( \partial \Omega_{cf} \) with \( \text{Im}(z_{cf}) > 0 \) and the system

\[
\begin{cases}
  s + \frac{f(s-c)}{1-2cs+c^2} + b - \frac{f}{1-c} = 0 \\
  \sqrt{(1-s^2)} \left(1 - \frac{f}{1-2cs+c^2}\right) - (1-b) = 0
\end{cases}
\]  

(4.1)
has a solution \((s, f)\) where \(s = \cos \zeta \in (0, c)\). Observe that \(f\) can be solved from the \(\text{A}rst\) equation of (4.1) in terms of \(s\), denoted by \(f_c(s)\),

\[
f_c(s) = \frac{(s + b)(1 - 2cs + c^2)(1 - c)}{(1 - s)(1 + c)}.
\]

Substituting (4.2) into the second equation of (4.1), we have

\[
g(s) := \sqrt{(1 - s^2)} \left(1 - \frac{(1 - c)(s + b)}{(1 - s)(1 + c)}\right) - (1 - b) = 0.
\]

It suffices to show that \(g(s)\) has a zero in \((0, c)\). Observe that \(g\) is continuous and differentiable on \((-1, 1)\) and

\[
g'(s) = \frac{2s^2 - 2s - 1 - b + c + bc}{(1 + c)(1 - s)\sqrt{1 - s^2}} = 0 \iff 2s^2 - 2s - 1 - b + c + bc = 0.
\]

Clearly, \(g'(s)\) has two real zeros:

\[
s_1 = \frac{1}{2} \left( \frac{1}{2} \sqrt{3 + 2b - 2c - 2bc} \right) \quad \text{and} \quad s_2 = \frac{1}{2} + \frac{1}{2} \sqrt{3 + 2b - 2c - 2bc}.
\]

It follows from \(0 < b < 1\) and \(0 < c \leq 1 - \sqrt{b}\) that

\[
1 < 3 - 2c + 2b(1 - c) = 3 + 2b - 2c - 2bc < 5,
\]

which implies that

\[-1 < s_1 < 0 \quad \text{and} \quad s_2 > 1.
\]

Then the equation \(g'(s) = 0\) has only one real solution in \((-1, 1)\). Notice that

\[
g(-1) = -(1 - b) < 0,
\]

\[
g(0) = -(1 - c) \frac{b}{1 + c} + b = \frac{2bc}{1 + c} > 0,
\]

\[
g(c) = \sqrt{(1 - c^2)} \left(1 - \frac{c + b}{1 + c}\right) - 1 + b = \frac{\sqrt{1 - c^2}(1 - b)}{1 + c} - (1 - b) < 0, \quad \text{and}
\]

\[
\lim_{s \to 1^-} g(s) = -\infty,
\]

and consequently, by Rolle's Theorem and the Intermediate Value Theorem, \(g(s) = 0\) has exactly two real solutions: one in \((-1, 0)\) and the other \(s^*\) in \((0, c)\). Thus, \(f^* = f_c(s^*)\) is the desired value of \(f\), which completes the proof of theorem. ■

Let \((s^*, f_c(s^*))\) be the solution of the system (4.1) for \(s^* \in (0, c)\) and \(f^* = f_c(s^*)\).

**Lemma 4.2** Let \(c \in (0, 1 - \sqrt{b})\). Then

\[
f^* \geq b(1 - c^2).
\]
Proof. Simplify the expression \( f_c(s) - b\left(1 - c^2\right) \) from (4.2) as follows:

\[
f_c(s) - b\left(1 - c^2\right) = \frac{(1-c)}{(1-s)(1+c)}((s+b)(1-2cs+c^2) - b(1-s)(1+c^2))
= \frac{(1-c)}{(1-s)(1+c)}(-2cs^2 + (1+b)(1+c^2)s - 2bc)
= \frac{-2c(1-c)}{(1-s)(1+c)}(s-u)(s-v)
\]

where

\[
u = \frac{1}{4c}\left((1+b)(1+c^2) - \sqrt{\Delta}\right) \quad \text{and} \quad v = \frac{1}{4c}\left((1+b)(1+c^2) + \sqrt{\Delta}\right)
\]

and

\[
\Delta = (1+b)^2(1+c^2)^2 - 16bc^2 = (1+b)^2(1+c^2)^2 - \left(2\sqrt{b}\right)^2(2c)^2 > 0.
\]

Now, it suffices to show \( s^* \in (u,v) \).

First rewriting \( g(s) = 0 \) in (4.3) and then simplifying the resulting equation, we have

\[(1+s)((1-s)(1+c) - (1-c)(s+b))^2 - (1-b)^2(1-s)(1+c)^2 = 0\]
or

\[
h(s) := \left(4cb + 4c^2b - 4cb^2\right) + (-2 + 2c^2b^2 + 2c^2 - 8cb + 2b^2) s
+ (-4cb + 4b - 4c)s^2 + 4s^3 = 0.
\]

Clearly, \( g(s) = 0 \) implies \( h(s) = 0 \). If both \( g \) and \( h \) have a unique zero in an open interval \((\alpha, \beta)\) and \( h(s_1) = 0 \) for \( s_1 \in (\alpha, \beta) \), then \( g(s_1) = 0 \).

Let

\[
u_1 = \frac{2bc}{1+b}.
\]

We show that \( s^* \in (u_1, c) \subset (u,v) \).

Since \( 0 < b < 1 \), \( 0 < u_1 < c \). Observe that

\[
h(-1) < 0,
\]

\[
h(0) = 4cb + 4c^2b - 4cb^2 > 0,
\]

\[
h(u_1) = 4c^2b\frac{(b-1)^2(cb^2 + b + c + 1)}{(1+b)^3} > 0,
\]

\[
h(c) = 2c(-1 + c)(c + 1)(b-1)^2 < 0, \quad \text{and}
\]

\[
h(1) > 0.
\]
So the cubic polynomial equation \( h(s) = 0 \) has three real solutions in intervals \((-1, 0)\), \((u_1, c)\) and \((c, 1)\), respectively. Since \( g(s) = 0 \) has a unique solution \( s^* \) in \((0, c)\) as shown in the proof of Theorem 4.1, \( s^* \in (u_1, c) \).

We prove that \((u_1, c) \subset (u, v)\) by showing the inequalities: \( u_1 > u \) and \( v > c \). For \( u \) and \( v \) given in (4.4),

\[
\begin{align*}
\frac{u_1 - u}{1 + b} &= \frac{2bc}{(1 + b) (1 + c^2) + \sqrt{\Delta}} - \frac{4bc}{(1 + b) (1 + c^2) + \sqrt{\Delta}} (1 + b) (1 + c^2) + \sqrt{\Delta} - 2 (1 + b) \\
&= \frac{2bc}{(1 + b) (1 + c^2) + \sqrt{\Delta}} \left( \sqrt{\Delta} - (1 + b) (1 - c^2) \right) \\
&= \frac{2bc}{(1 + b) (1 + c^2) + \sqrt{\Delta}} \left( \sqrt{\Delta} - (1 + b) (1 - c^2) \right) \\
&= \frac{2bc}{(1 + b) (1 + c^2) + \sqrt{\Delta}} \left( \sqrt{\Delta} + (1 + b) (1 - c^2) \right) > 0 
\end{align*}
\]

and

\[
\begin{align*}
\frac{v - c}{1 + b} &= \frac{4bc}{(1 + b) (1 + c^2) + \sqrt{\Delta}} - \frac{c}{(1 + b) (1 + c^2) + \sqrt{\Delta}} (4b + \sqrt{\Delta} - (1 + b) (1 + c^2)) > 0 
\end{align*}
\]

since

\[
\Delta - ((1 + b)^2 (1 + c^2) - 4b)^2 = 8b (1 - c^2)(1 - b) > 0
\]

\[
\square
\]

**Lemma 4.3** Let \( b \geq \sqrt{5} - 2 \) and \( c \in \left(0, \frac{1 - b}{2}\right)\). Then

\[
f^* \leq \min \{ b(1 + c^2), (1 - c)^2 \}.
\]

**Proof.** Let \( c \in \left(0, \frac{1 - b}{2}\right)\) and \( b \geq \sqrt{5} - 2 \). The inequalities (a) \( f^* \leq b(1 + c^2) \) and (b) \( f^* \leq (1 - c)^2 \) are shown separately as follows.

(a) \( f^* < b \leq b(1 + c^2) \).

We show \( f^* < b \) for \( c \) in \( \left(0, 1 - \sqrt{b}\right) \) which contains \( \left(0, \frac{1 - b}{2}\right) \). From Theorem 1.4, \( z_{c,f^*} = z_c \) would never be the vertex \( d + ia(1 - b) \), which leads to a contradiction to the properties of \( f^* \). Assume that \( f^* > b \). Let \( \Omega_1 \) be the ellipse passing through the point \( \Psi(1) \) with a major axis of length \( 2a(1 + f^*) \) and a minor axis of length \( 2a(1 - f^*) \). Clearly, \( \partial \Omega_1 \) would be below \( \partial \Omega \) in the I\text{st} quadrant of \( \partial \Omega \).
By Theorem 1.4, the curve $\partial \Omega_{f^*}$ would be below $\partial \Omega_1$ in the first quadrant of $\partial \Omega$. Then $\partial \Omega_{f^*}$ could never pass the vertex $d + ia(1 - b)$ of $\partial \Omega$, which contradicts the result in Theorem 4.1. Therefore $f^* < b \leq b(1 + c^2)$.

(b) $f^* \leq (1 - c)^2$.

Observe from (4.2) that

$$f_c(s) - (1 - c)^2 = \frac{(1 - c) ((s + b)(1 - 2cs + c^2) - (1 - s)(1 - c^2))}{(1 - s)(1 + c)} =: \frac{(1 - c)}{(1 - s)(1 + c)} P(s) < 0$$

for $s \in (0, c)$ because of the following reasons:

(i) $P(s)$ is quadratic and concave down;

(ii) $P(c) = (c + b)(1 - c^2) - (1 - c)(1 - c^2) = (1 - c^2)(2c + b - 1) < 0$ since $c < \frac{1 - b}{2}$ implies $2c + b - 1 < 0$; and

(iii) $P(1) = (1 + b)(1 - c^2) > 0$.

Since $s^* \in (0, c)$, $f^* \leq (1 - c)^2$.

\[\Box\]

**Theorem 4.4** For any $c \in (0, 1 - \sqrt{b}]$ and $b \geq \sqrt{5} - 2$, there exists an $f^* \in \left[b(1 - c^2), \min\{b(1 + c^2), (1 - c)^2\}\right]$ such that the point $z_{cf^*} = d + ia(1 - b)$ is a unique intersection point of $\partial \Omega$ and $\partial \Omega_{f^*}$ with $\text{Im}(z_{cf^*}) > 0$.

**Proof.** It follows directly from Lemma 4.1, 4.2 and 4.3. \[\Box\]

5 The Superiority of the New (2,2)-step Method

As shown in Section 4, a $\Psi_{cf}(w)$ can be constructed when (2.5) holds and $b \geq \sqrt{5} - 2$ so that all vertices of the given ellipse $\Omega$ and $\sigma(T)$ are contained in $\Omega_{cf}$ for some $c \in \left(0, \frac{1 - b}{2}\right)$. In this section, the properties of $\kappa(\Omega_{cf})$ of (2.4) are studied and the superiority of the (2,2)-step iterative method generated by $\Psi_{cf}$ is derived.

**Lemma 5.1** $\kappa(\Omega_{cf})$ is monotonically decreasing with respect to $f$ for $f \in (- (1 - c^2), (1 - c)^2)$ and $c \in (0, 1 - \sqrt{b}]$. 


**Proof.** Define for $|w| > 1$

$$A(c, f, w) := \Psi_{cf}(w) - 1 = a \left( w + \frac{f}{w - c} \right) + (d - 1) + a \left( b - \frac{f}{1 - c} \right)$$

and

$$\rho := \rho(c, f) := \Psi_{cf}^{-1}(1). \quad (5.1)$$

Then

$$A(c, f, \rho) = A(c, f, \rho(c, f)) = 0.$$ 

As given in Section 1, $\Psi_{cf}(w)$ is an one-to-one and onto mapping with properties at $\infty$: $\Psi_{cf}(\infty) = \infty$ and $\Psi_{cf}'(\infty) > 0$. So, $\rho > 1$. Since $f \leq (1 - c)^2 < (\rho - c)^2$ and $\rho > 1$,

$$\frac{\partial A}{\partial \rho} = a \left( 1 - \frac{f}{(\rho - c)^2} \right) \neq 0$$

and

$$\frac{\partial \rho}{\partial c} = -\frac{\partial A}{\partial c} \frac{\partial A}{\partial \rho} = -\frac{af}{(\rho - c)^2} - \frac{af}{(1 - c)^2} = -\frac{f \left( (1 - c)^2 - (\rho - c)^2 \right)}{(\rho - c)^2 - f} \frac{1}{(1 - c)^2} > 0,$$

$$\frac{\partial \rho}{\partial f} = \frac{\partial A}{\partial f} \frac{\partial A}{\partial \rho} = -\frac{a}{(\rho - c)(1 - c)} - \frac{a - \rho}{(\rho - c)^2} = -\frac{(1 - \rho)(\rho - c)}{(\rho - c)^2 - f} > 0.$$ 

Let $d = [\cos \theta, \sin \theta]$ for $0 \leq \theta \leq \frac{\pi}{2}$ be a unit vector. Observe that the directional derivative of $\rho(c, f)$ along $d$,

$$D_d \rho(c, f) = \frac{\partial \rho}{\partial c} \cos \theta + \frac{\partial \rho}{\partial f} \sin \theta > 0$$

for all unit directions $d$. Then the function $\rho(c, f)$ is monotonically increasing for $c \geq 0$ and $f \geq 0$, and therefore

$$k(\Omega_{cf}) = \frac{1}{\Psi_{cf}^{-1}(1)} = \frac{1}{|\rho(c, f)|} \quad (5.2)$$

is monotonically decreasing. ■

**Theorem 5.2** There exists $f_1 \in (0, b(1 - c))$ such that $\kappa(\Omega_{cf_1}) = \kappa(\Omega)$ for a given $c \in (0, 1 - \sqrt{b})$. Moreover, $f_1 < f^*$ and $\kappa(\Omega_{cf_1}) < \kappa(\Omega)$.

**Proof.** It follows from (5.1) and (5.2) that

$$\kappa(\Omega_{cf_1}) = \kappa(\Omega) \text{ if and only if } \rho(c, f_1) = \rho(0, b). \quad (5.3)$$
Observe that
\[ \rho(c, f) = \frac{1}{2a} \left( ac - d + 1 - a \left( b - \frac{f}{1-c} \right) + \sqrt{\Delta} \right) \quad \text{and} \]
\[ \rho(0, b) = \frac{1}{2a} \left( -d + 1 + \sqrt{(d-1)^2 - 4a^2b} \right), \]
where
\[ \Delta = \left( ac + d - 1 + a \left( b - \frac{f}{1-c} \right) \right)^2 - 4a^2f. \]

Solving directly from (5.3), \( f_1 \) can be expressed as
\[ f_1 = \frac{(1 - c) c \left( -1 + d + 2ab + \sqrt{(d^2 - 2d + 1 - 4a^2b)} \right)}{-1 + 2a + d - \sqrt{(d^2 - 2d + 1 - 4a^2b)}} + b(1 - c). \]

Further simplifying it, we have
\[ f_1 = (1 - c) \frac{c \left( -1 + d + \sqrt{(d^2 - 2d + 1 - 4a^2b)} \right) + 2ab}{2a} \]
\[ = b(1 - c) \left( 1 - \frac{2ac}{(1 - d) + \sqrt{(d-1)^2 - 4a^2b}} \right) \]
\[ < b(1 - c) < b \left( 1 - c^2 \right). \]

As shown in Lemma 4.2, \( f^* \geq b \left( 1 - c^2 \right) \). Hence, \( f_1 < f^* \). It follows from Theorem 5.1,
\[ \kappa(\Omega_{c,f^*}) < \kappa(\Omega_{c,f_1}) = \kappa(\Omega). \]

Now we go back to the question raised in Section 1: When the ratio of the minor axis to the major axis of an optimal ellipse is less than or equal to the golden ratio, i.e., \( b \geq \sqrt{5} - 2 \), can the asymptotic rate of convergence of the optimal Chebyshev method be improved by a (2,2)-step method generated by other than \( \Psi_c \)? Clearly, the answer is yes and the detail is given by the following theorem.

**Theorem 5.3** If \( b \geq \sqrt{5} - 2 \) and (2.5) holds, then there is a (2,2)-step iterative method generated by \( \Psi_{c,f} \) for \( c \in \left( 0, \frac{1-b}{2} \right) \) such that its asymptotic rate of convergence improves the one of the optimal Chebyshev method.

**Proof.** It follows from Theorem 4.1 that for any \( c \in \left( 0, \frac{1-b}{2} \right) \), there exists \( f^* = f_c(s^*) \) such that \( \partial \Omega_{c,f^*} \) passes through the point \( d + ia(1-b) \), the vertex of \( \partial \Omega \) on the upper half-plane. Moreover, (3.7) holds by Lemma 4.2 and Lemma 4.3. Therefore,
\[ \kappa(\Omega_{c,f^*}) < \kappa(\Omega). \]
from Theorem 5.2. Since no eigenvalue of $T$ belongs to $\{\Psi(e^{it})| 0 < t < \pi/2\}$, there exists a $c_1 \in \left(0, \frac{1-b}{2}\right)$ such that $\sigma(T) \subset \Omega_{cf} \cap \Omega$ for any $c \in (0, c_1)$. Thus the asymptotic rate of convergence of the (2,2)-step method generated by $\Psi_{cf}$, which is equal to $\kappa(\Omega_{cf})$, improves the one of the optimal Chebyshev method, which is equal to $\kappa(\Omega)$. ■

6 Examples

In this section, two examples are given to illustrate the superiority of the new type (2,2)-step iterative method developed in this paper in the case where $b \geq \sqrt{5} - 2$. Moreover, we compare the new type (2,2)-step method with Chebyshev, restarted GMRES [13], BICGSTAB [15] and QMR [4] methods. The M-\texttt{iles} in MATLAB 6.1 for the last three iterative algorithms are used in computation. For simplicity, no preconditioner is used. The exact solutions for both examples are $x^* = [1, ..., 1]^T$. The initial estimate $x_0 = [0, ..., 0]^T$ is chosen. The stopping criteria is defined by $||r_k||_2 \leq 10^{-8}||r_0||_2$, where $r_k$ is the $k$th residual vector.

Example 6.1 Consider the boundary value problem of the form [14]

$$-\Delta u + 2p_1 u_x + 2p_2 u_y - p_3 u = F, \quad (x, y) \in (0, 1) \times (0, 1) =: G,$$

$$u(x, y) = 0, \quad (x, y) \in \partial G$$

where $p_1, p_2, p_3$ and $F$ are functions of $x$ and $y$. Using the standard $N$-\texttt{ove-point} discretization to approximate the Laplacian $\Delta$, a system of linear equation

$$x = Tx + c$$

with $N$ unknowns is obtained. In this example:

$$p_1 = 13(x - 2y), \quad p_2 = 7(2x - y - 3), \quad p_3 = 120, \quad \text{and} \quad N = 400.$$ 

The spectrum of $T$ and $\partial \Omega$ are plotted in Figure 1. The conditions in (2.5) for $\sigma(T)$ are satisfied. Since $b \geq \sqrt{5} - 2$, the Chebyshev iterative method corresponding to $\Psi$ of (1.2) may not be improved by a (2,2)-step iterative method generated by $\Psi_c$. However, a (2,2)-step iterative method yielded by $\Psi_{cf}$ can be used. Choosing $c = 0.3$, $f = 0.35453$ is calculated by (4.2) for $s \in (0, c)$ being the unique real solution of $g(s) = 0$. $\partial \Omega_{cf}$ is also plotted in Figure 1. It is easy to see that the upper vertex of $\partial \Omega$ is an intersection point of $\partial \Omega$ and $\partial \Omega_{cf}$ and $\sigma(T) \subset \Omega_{ef}$.

Figure 2 shows that the (2,2)-step iterative method generated by $\Psi_{ef}(w)$ is better than Chebyshev method while GMRES(20) method does not converge. However, the new type (2,2)-step iterative method is less efficient than BICGSTAB and QMR methods.
Figure 1: ... $\sigma(T)$, $x$ the graph of $\partial\Omega$, -- the graph of $\partial\Omega_{cf}$

Figure 2: Comparison of the new type (2,2)-step, Chebyshev, GMRES(20), BICGSTAB and QMR
Example 6.2 This example is very similar to that of [11]. Computation results show that the new type (2,2)-step method is superior to not only Chebyshev method but also other iterative methods. Consider a banded Toeplitz matrix $A = I - T$, where $T$ is given by

$$ T = \begin{bmatrix} -4 & -0.8 & 1.8 \\ 1 & -4 & -0.8 \\ 1 & 1 & -4 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{512 \times 512}. $$

The spectrum of $T$, the ellipse $\partial \Omega$ and the curve $\partial \Omega_{cf}$ for $c = 0.2$ and $f = 0.28088$ are plotted in Figure 3 where $f$ is calculated in the same way as in Example 6.1. Figure 4 shows that the new type (2,2)-step iterative method is better than Chebyshev, GMRES(20), BICHSTAB and QMR methods.
References


